Dynamic Hedging of Real Wealth Risk

STEFAN SCHUBERT

UDO BROLL

Dresden Discussion Paper in Economics No. 01/05
Address of the author(s):

Stefan Schubert  
University of Munich  
Eduard-Spranger-Strasse 8  
80935 Munich  
Germany  
e-mail: stefan.schubert@bayern.landtag.de

Udo Broll  
Dresden University of Technology  
Faculty of Business Management and Economics  
01062 Dresden  
Germany  
e-mail: Udo.Broll@mailbox.tu-dresden.de

Editors:
Faculty of Business Management and Economics, Department of Economics

Internet:
An electronic version of the paper may be downloaded from the homepage:  
http://rcswww.urz.tu-dresden.de/wpeconomics/index.htm  
English papers are also available from the SSRN website:  
http://www.ssrn.com

Working paper coordinator:
Oliver Greßmann  
e-mail: wpeconomics@mailbox.tu-dresden.de
Dynamic Hedging of Real Wealth Risk

Stefan Schubert
University of Munich
80935 Munich
stefan.schubert@bavern.landtag.de

Udo Broll
Dresden University of Technology
01062 Dresden
Udo.Broll@mailbox.tu-dresden.de

Abstract:
International and national investors are often exposed to real wealth risks, stemming from volatile asset prices and inflation uncertainty, making it difficult to stabilize consumption patterns. However, investors can enter futures markets to hedge against these risks. The paper develops a simple continuous-time dynamic model, where the evolution of asset price, price level and futures price and hence real wealth is stochastic. For a risk averse investor, optimal consumption and hedging strategy are derived and discussed. It is shown that hedging increases the investor’s wellbeing in terms of intertemporal utility of consumption.

JEL-Classification: F21, F31

Keywords: wealth, asset price, dynamic hedging, optimum consumption
1. Introduction

The importance of risk management has inspired numerous contributions to the theory of investment and consumption under uncertainty (see, e.g., Bodie (1976), Holthausen (1979), Benninga, Eldor, and Zilcha (1983), Kawai and Zilcha (1986); Lence (1995), Adam-Müller (2000), to name just a few). Most of the literature on economic risk and risk aversion dealing with investment, consumption and hedging decisions has incorporated the assumption that investors are concerned about random nominal wealth denominated in one currency. However, the investor’s wealth may also change when domestic prices change, for example to an unexpected inflation rate (see, e.g., Blanchard (2003)). The holder of an asset should not neglect inflation risk, since her consumption opportunity set also changes. Therefore, the analysis should be imbedded in a framework of inflation risk (see Battermann and Broll (2001, 2004), Elder (2004), Schubert and Broll (2004)).

In this paper we discuss the impact of asset price volatility on consumption and the hedging concepts that have been suggested in the literature. In contrast to most of the literature, the study develops a simple continuous-time dynamic consumption and hedging model, where the evolution of asset price and price level and therefore real wealth of the investor is stochastic (for continuous-time dynamic hedging models, see, e.g., Briys and Solnik (1992), Tong (1996), Lioui and Poncet (2000)). Given risk aversion of the investor, optimal consumption and hedging strategy are derived and discussed. We demonstrate that a dynamic hedge increases the investor’s wellbeing in terms of intertemporal utility of consumption.

Inflation causes money to decrease in value, and does so whether the money is invested or not. Real wealth is therefore uncertain for two reasons: The value of accumulated assets is uncertain, and the overall price level is uncertain. However, the investor can step into the futures market and try to reduce the volatility of real wealth by hedging against asset price fluctuations and the correlated change in the overall price level (see Schubert and Broll (2004)). Our discussion illustrates a typical use of futures contracts, and not on how to price these futures contracts. The latter would involve the forecasting of future inflation, a topic outside the scope of this study.
We demonstrate that the optimal dynamic hedging strategy can be decomposed in a variance minimizing component and in a speculative component, depending on the investor’s degree of relative risk aversion. It is shown that the optimal consumption-wealth-ratio may be greater or smaller as in case without the hedging instrument, depending on the magnitude of relative risk aversion. However, in any case we show that the investor’s overall wellbeing is greater in case with hedging.

The paper is structured as follows: Section 2 sets up the model, in section 3 the investor’s optimization problem and the optimal consumption and hedging strategies are explained. Section 4 briefly discusses some simple cases. In section 5, we employ the isoelastic utility function to gain further insight in the effects of hedging on the investor’s objective. Section 6 summarizes the results. An appendix contains the necessary calculations.

2. The Model

Consider a risk-averse individual, called investor, who earns no labor income and who has accumulated real wealth

\[ W = \frac{SA}{P}, \]  

(1)

where \( S \) denotes the spot market price of assets, \( A \) is the stock of assets accumulated, and \( P \) represents the overall price level. Both \( S \) and \( P \) are stochastic. Applying Itô’s lemma, the change in the investor’s real wealth can be written as

\[ dW = \left[ \frac{dS}{S} - \frac{dP}{P} \cdot \frac{dS}{S} \cdot \frac{dP}{P} + \left( \frac{dP}{P} \right)^2 \right] W + \frac{S}{P} dA. \]  

(2)

The investor can take a short or long position on the futures market of volume \( H \) (a positive/negative \( H \) denotes a short/long position). To enter the futures market costs nothing, but from thereon the investor’s margin account has to be continuously adjusted due to changes in the futures price \( F \). Moreover, the investor can consume at the rate \( Cdt \), where \( C \) denotes the rate of real consumption. \( C \) can be broadly defined as a composite good. Thus, the last term of eq. (2), the change of the value of accumulated assets at any instant, given \( S/P \), is given by

\[ S \cdot dA = -PCdt - H(dF). \]
Introducing the hedge ratio \( h \equiv \frac{HF}{PW} \), the change of the real value of accumulated assets (given \( S \) and \( P \)) can be written as

\[
\frac{S \cdot dA}{P} = -C dt - h \frac{dF}{F} W,
\]

which says that beside price changes, real wealth changes due to real consumption and hedging. Hence, real wealth accumulation (2) becomes

\[
dW = \left[ \frac{dS}{S} - \frac{dP}{P} - \frac{dS dP}{SP} + \left( \frac{dP}{P} \right)^2 \right] - h \left( \frac{dF}{F} \right) W - C dt.
\]

For the sake of simplicity, we assume that the evolution of \( S \), \( P \), and \( F \) can be described by geometric Brownian motions:

\[
\begin{align*}
\frac{dS}{S} &= \mu_s dt + \sigma_s dz_s \\
\frac{dP}{P} &= \mu_p dt + \sigma_p dz_p \\
\frac{dF}{F} &= \mu_f dt + \sigma_f dz_f
\end{align*}
\]

where \( \mu_i \) represents the expected instantaneous rate of change and \( \sigma_i \) is the volatility parameter. For example, \( \mu_p \) can be interpreted as trend inflation rate, which itself is determined by the growth rate of money supply. \( \mu_f \) can be understood as the expected rate of change of the futures price. The \( dz_i \) are standard Wiener processes with mean zero and instantaneous variance \( dt \). Note that \( \sigma_i dz_i \sigma_j dz_j = \sigma_i \sigma_j \rho_{ij} dt \equiv \sigma_{ij} dt \), where \( \sigma_{ij} dt \) denotes the instantaneous covariance between prices \( i \) and \( j \), and \( \rho_{ij} \) is the correlation coefficient between the two standard Wiener processes \( dz_i \) and \( dz_j \). Note further that \( \mu_i dt \sigma_j dz_j = o(dt) \).

Inserting (4) into real wealth accumulation (2'), dropping terms of higher order than \( dt \), and collecting terms we obtain

\[
dW = \left[ \mu_s - \mu_p - \sigma_{sp} + \sigma_s^2 dt + (\sigma_s dz_s - \sigma_p dz_p) - h(\mu_p dt + \sigma_p dz_p) \right] W - C dt
\]

This is the stochastic (real) wealth accumulation equation.
3. Optimal consumption and hedging strategy

The investor’s objective is to maximize the expected present value of utility of consumption over his planning horizon \( T \)

\[
V(W,t) = \max_{C,h} E_0 \int_0^T U(C) e^{-\beta t} dt
\]

subject to (5). The instantaneous utility function \( U(C) \) with \( U_C > 0, U_{CC} < 0 \) is assumed to be time separable. \( \beta \) denotes the investor’s constant rate of time preference. Eq. (6) shows that the investor ultimately cares about real consumption; she tries to reduce the volatility of wealth as expressed by (5) via choosing an appropriate hedge ratio \( h \), allowing thus for a more smoothly consumption profile.

Applying Itô’s lemma to the value function \( V(W,t) \), as shown in the appendix the Bellman equation to problem (6) is

\[
\beta V(W,t) = \max_{C,h} \left\{ U(C) + V_t + V_W \left( \mu_S - \mu_p - \sigma_{SP} + \sigma_p^2 \right) - h \mu_F \right\} W - V_{WW} C
\]  
\[+ \frac{1}{2} V_{WW} \left[ \sigma_S^2 + \sigma_p^2 - 2 \sigma_{SP} - 2h(\sigma_{SF} - \sigma_{PF}) + h^2 \sigma_F^2 \right] W^2 \]

Maximization of the right hand side of (7) implies the following first order conditions

\[ U_C(\hat{C}) = V_{W}(W,t) \]  
\[ \hat{h} = \frac{\sigma_{SF} - \sigma_{PF}}{\sigma_F^2} + \frac{\mu_p V_W(W,t)}{W \sigma_F^2 V_{WW}(W,t)} \]

where \(^\wedge\) denotes optimized value. Eq. (8a) is a standard optimality condition and equates the marginal utility of consumption to the marginal utility of wealth, \( V_W \). Solving this equation for \( C \) yields the investor’s rate of real consumption. Eq. (8b) describes the investor’s optimal hedge ratio \(( \hat{h} > 0/\hat{h} < 0 \) implies a short/long hedge) and merits further comment:

The first term on the right hand side of (8b) represents the variance minimizing hedge ratio and can be understood as follows. On the one hand, hedging is used to reduce the variance of the time path of real wealth. On the other hand, hedging introduces an additional risk via the volatility of the futures price. Thus, the higher it’s variance \( \sigma_F^2 \), the less hedging is able to reduce fluctuations of wealth, and thus the lower the optimal hedge ratio is. The variance minimizing hedge ratio depends positively on the covariance between the spot price \( S \) and the
futures price $F$. In case of a positive covariance, on average $S$ and $F$ move in the same direction, and thus via margin account adjustments a short hedge works against real wealth increases due to increases in the nominal value of assets $SA$. The opposite is true for a positive covariance between the future price and the price level $P$. To stabilize real wealth, an increase in the price level and thus a reduction of real wealth requires a long hedge, because on average losses of wealth’s purchasing power are offset by increases on the margin account. This explains the minus-sign of the $\sigma_{PF}$-term. In the special case $\sigma_{SF} = \sigma_{PF}$, these two hedging components cancel out, and the variance minimizing hedge becomes zero. Intuitively, on average $S$ and $P$ then move exactly and proportionally in tandem, hence real wealth is automatically stabilized.

The second term on the right hand side of (8b) is the speculative component of the hedge ratio. A rational investor will deviate from the variance minimizing hedge ratio to the extent hedging provides an additional source of profit. If, e. g., $\mu_F > 0$, the futures price is expected to increase over time, thus a long position gives rise to expected profits via the margin account. However, in contrast to the variance minimizing hedge ratio, the speculative component depends not only on the variance of the futures price, but also to the investor’s relative risk aversion $\frac{\delta W}{W} > 0$. Thus, the higher $\sigma_F^2$ and the greater relative risk aversion, the smaller the speculative hedge ratio becomes.

The overall hedge ratio is thus a combination of variance minimizing and speculative components. Which of these components dominate depends on the volatility parameters $\sigma_{SF}, \sigma_{PF},$ and $\sigma_F^2$, on the expected rate of change $\mu_F$, and on the investor’s relative risk aversion. In general, relative risk aversion is a function of wealth and time, hence the optimal hedge ratio is time-dependent and will change over time, i. e., the hedge ratio is dynamic. It is important to recognize that there is scope for hedging even in case of a deterministic asset price $S$ due to price level uncertainty.

4. Simple cases

Let us briefly consider the simple special case of a deterministic evolution of the price level by setting $\sigma_p^2 = \sigma_{SP} = \sigma_{PF} = 0$. The optimal hedge ratio becomes then
\[ \hat{h} = \frac{\sigma_S}{\sigma_F^2} + \frac{\mu_F V_W(W,t)}{W \sigma_F^2 V_{WW}(W,t)}. \]

Whereas the speculative component remains unchanged, the variance minimizing hedge rate simplifies, as it depends only on the covariance between the spot asset price and the future price (and, of course, on its variance). A positive correlation implies thus a short hedge to minimize the variance of wealth. If in addition the asset price and the future price are perfectly correlated, i.e., if they obey the same Wiener process \( \sigma_S dz_S = \sigma_F dz_F \) and thus \( \sigma_{SF} = \sigma_F^2 \), we get the well-known standard result (see, e.g., Kawai and Zilcha (1986), Lence (1995), Adam-Müller (2000))

\[ \hat{h} = 1 + \frac{\mu_F V_W(W,t)}{W \sigma_F^2 V_{WW}(W,t)}. \]

The variance minimizing component equals then unity. Since in this case the spot asset price and the future price exactly move in tandem, a full short hedge eliminates any wealth fluctuations. This means that the investor incurs a short position at the futures market whose value equals the nominal value of her wealth. However, a rational investor will deviate from a full hedge to take advantage of the expected change in the future price to the extent of her relative risk aversion.

Contrarily, if the evolution of the asset’s spot price \( S \) but not of the price level is deterministic, \( \sigma_{SF} = 0 \) and the variance minimizing hedge ratio in equation (8b) simply becomes \( -\sigma_{PF} / \sigma_F^2 \). A positive covariance between the price level and the futures price requires then a long hedge to minimize the variance of wealth accumulation.

The ultimate purpose of hedging is to stabilize the evolution of real wealth, enabling the investor to smooth her consumption profile and to increase her intertemporal utility \( V(W,t) \).

To see this more formally, in the next section we shall consider a particular utility function.

5. The effects of hedging - a closer look

To get more insight into the hedging strategy and its effects, and for analytical convenience, from now on we shall assume that the investor’s utility function is isoelastic, and that her planning horizon is infinite. Her maximization problem becomes thus
\[ V(W) = \max_{C,h} \left\{ \gamma \frac{1}{\gamma} C'^2 e^{-\beta t} dt, \quad -\infty < \gamma < 1 \right\} \]

where \( \gamma = 1 \) denotes the investor's degree of relative risk aversion. In case of \( \gamma = 0 \), the utility function is logarithmic. Since (i) the planning horizon is infinite, (ii) the utility function is additively separable in time, and (iii) the involved stochastic processes (4) do not directly depend on time, the value function \( V(\cdot) \) can be expressed in terms of wealth \( W \) solely.\(^1\) As shown in the appendix, the value function, optimal consumption, and the hedge ratio become

\[
V(W) = \frac{1}{\gamma} \left( \frac{\hat{C}}{W} \right)^{\gamma-1} W^\gamma 
\]

\[
\frac{\hat{C}}{W} = \frac{\beta - \hat{a} \gamma - \hat{b} \gamma (\gamma - 1)/2}{1 - \gamma} 
\]

\[
\hat{h} = \frac{\hat{\sigma}_{S}^2 - \hat{\sigma}_{PF}^2}{\hat{\sigma}_{F}^2} + \frac{\mu_F}{\hat{\sigma}_{F}^2 (\gamma - 1)} 
\]

where:

\[
\hat{a} \equiv \mu_s - \mu_p - \sigma_{SP}^2 - \hat{h} \mu_F, \quad \hat{b} \equiv \sigma_{S}^2 + \sigma_{P}^2 - 2\sigma_{SP} - 2\hat{h}(\sigma_{SP} - \sigma_{PF}) + \hat{h}^2 \sigma_{F}^2 
\]

\( \hat{a} \) describes the expected real rate of return on wealth, and \( \hat{b} \) measures the variance of the Wiener process for real wealth accumulation,\(^2\) both when the optimal hedging policy (9c) is chosen. The following observations can be made:

Eq. (9c) confirms that the optimal hedge ratio is the smaller (in absolute value) the greater the investor's relative risk aversion (\( \gamma - 1 \)), and the higher the variance of the futures price. It is worth to point out that in case of an infinite planning horizon combined with an isoelastic utility function and stochastic processes described by geometric Brownian motions, the optimal hedge ratio is time-invariant, i.e., the hedge ratio is static. From (9b) in combination with the time-invariant optimal hedge ratio, entering \( \hat{a} \) and \( \hat{b} \), it follows that the optimal consumption-wealth-ratio is time-invariant, too.

Depending on \( \gamma > 0 \), from (9b) we see that an increase in the expected rate of return on real wealth \( \hat{a} \) increases or lowers the consumption-wealth-ratio. There are two opposite effects at

\(^1\) See, e.g., Dixit and Pindyck (1994), p. 107. The Bellman differential equation is then an ordinary one, allowing to solving it analytically.

\(^2\) The part of the stochastic wealth accumulation equation not concerning consumption can thus be written as \( dW = \hat{a}Wdt + \hat{b}dw \), where \( dw \) is a Wiener process with zero mean and instantaneous variance \( \hat{b}dt \).
work. First, according to the income effect, a higher $\hat{a}$ increases consumption, thus increasing $C/W$. Second, due to the substitution effect, a higher expected rate of return encourages wealth accumulation, thus lowering $C/W$. Which effect dominates depends on $\gamma$. We also see that the expected real rate of return $\hat{a}$ depends on the optimal hedging strategy and on the futures price’s expected growth rate, i.e. on $(-\hat{h}\mu_r)$. A short hedge ($\hat{h} > 0$) in combination with a positive expected growth rate of the futures price ($\mu_f > 0$) lowers $\hat{a}$ because of expected losses, since an investor incurring a short position has to pay for margin account adjustments when the futures price is rising. For a long hedge in combination with a negative expected growth rate rate, the investor has to make payments to his margin account for a declining futures price, thus also leading to expected losses. On contrast, if the futures’ expected growth rate and the hedge ratio have opposite sign, the investor expects additional profits from hedging, increasing hence the expected rate of return on real wealth.

Similar but opposite income and substitution effects could be detected for the variance $\hat{b}$. A higher variance exercises a negative income effect, thus lowering $C/W$, whereas its positive substitution effect stimulates consumption. Which effect dominates depends again on $\gamma$. The optimal hedging strategy itself has two opposite effects on the variance of real wealth: (i) the variance minimizing component of $\hat{h}$ lowers the variance by $-(\sigma_{SF} - \sigma_{PF})^2 / \sigma_r^2$, whereas (ii) the speculative component unambiguously increases the variance by $\mu_f^2 / (\sigma_r^2(\gamma - 1)^2)$. The overall effect on the variance is therefore unclear.

Inserting (9c) into (9b) and simplifying, we get for the optimal consumption-wealth-ratio

$$\frac{\hat{C}}{W} = \frac{1}{1-\gamma} \left\{ \beta - \gamma(\mu_S - \mu_P - \sigma_{SP} + \sigma_S^2) - (1/2)\gamma(\gamma - 1)(\sigma_S^2 + \sigma_P^2 - 2\sigma_{SP}) + (1/2)\gamma(\gamma - 1)\sigma_S^2\hat{h}^2 \right\}$$

Since the first two terms in accolades can be identified as the optimal consumption-wealth-ratio without hedging possibilities, i.e. $\left(\hat{C}/W\right)_{\text{without hedging}}$, we may equally write

$$\frac{\hat{C}}{W} = \left(\frac{\hat{C}}{W}\right)_{\text{without hedging}} + \frac{1}{1-\gamma} (1/2)\gamma(\gamma - 1)\sigma_S^2\hat{h}^2$$  \hspace{1cm} (9b’)

Looking at (9b’), we note that the consumption-wealth-ratio may be greater or lower than in the case without hedging. Again, this depends on $\gamma$. In case of the logarithmic utility function, $\gamma = 0$, and the consumption-wealth-ratio is unaffected from hedging and is given by $\hat{C}/W = \beta$. It is important to note that it is not the investor’s objective to maximize utility by...
means of choosing the highest possible consumption-wealth-ratio. Ultimately, the investor only cares about utility of real consumption $C$. Most important, given initial real wealth $W(0)$ we can infer from (9a) in combination with (9b') that regardless of the concrete value of $\gamma \in (-\infty,1)$, at time zero hedging always leads to an intertemporal utility that is higher as in case without hedging.\(^3\)

We can thus summarize that in general the introduction of hedging against inflation risk increases the investor’s wellbeing, i.e.

$$V(W(0)) > V(W(0))_{\text{without hedging}}$$

(10)
as long as $\hat{h} \neq 0$.

Let us finally briefly discuss the effects of changes in the trend inflation rate $\mu_p$ and in the inflation volatility $\sigma^2_p$, which, e.g., may be caused by changes in monetary policy. An increase in $\mu_p$ clearly lowers the expected rate of return on real wealth $\hat{a}$, as can be seen from eq. (9d). An increase in $\sigma^2_p$ results in a higher variance parameter $\hat{b}$ of real wealth accumulation and, perhaps a little surprisingly, raises expected rate of return on real wealth. The reason is that due to the convexity of real wealth $W \equiv AS/P$ in $P$, an increase of $\Delta P$, say, reduces real wealth by less than a fall of magnitude $-\Delta P$ increases real wealth, hence real wealth increases on average. As we have already mentioned, changes of $\hat{a}$ and $\hat{b}$ cause income and substitution effects, and their overall effects on the consumption-wealth-ratio depend on the elasticity of utility $\gamma$. Most interestingly, the optimal hedging strategy $\hat{h}$ is unaffected as long as only $\mu_p$ and $\sigma^2_p$ change, since they do not appear in (9c). This does not mean that the investor will not change his futures position, however, since a constant hedge ratio implies that any change in the nominal value of wealth $PW$ has to be exactly offset by a proportional change in the futures position $HF$. Note that this result holds as long as the investor’s relative risk aversion is constant. Otherwise, changes in real wealth change the speculative component of the hedge ratio. Changes of monetary policy may result in changes of the other drift and volatility parameters, too, exercising further effects on $\hat{a}$ and $\hat{b}$, and then also on the optimal hedging strategy.

\(^3\) Note that at time zero wealth $W(0)$ is the same regardless of hedging possibilities, since signing a futures contract costs nothing.
6. Conclusion

In a simple continuous time framework where an investor faces real wealth risk, we derived the investor’s optimal consumption and hedging strategy. There, we demonstrated that the optimal hedging strategy can be decomposed in a preference free variance minimizing component and in a speculative component, depending on the investor’s degree of relative risk aversion. We discussed the interaction between the optimal hedge ratio, the covariance, the expected growth rate of the futures price, the volatility on the futures market, and the sign the optimal hedge ratio. In case of an infinite planning horizon, we calculated that the optimal consumption-wealth-ratio may be greater or smaller as in case without the hedging instrument, depending on the magnitude of relative risk aversion. However, in any case we showed that the investor’s overall wellbeing is greater in case with hedging.
APPENDIX

1. Derivation of the Bellman equation

The Bellman equation of the optimization problem is

\[
\beta V(W,t) = \max_{C,h} \left[ U(C) + \frac{E(dV(W,t))}{dt} \right]
\]

where \(dV(W,t)\) is calculated using Itô’s lemma:

\[
dV(W,t) = V_t dt + V_w dW + \frac{1}{2} V_{ww} (dW)^2 + o(dt)
\]

where \(dW\) is defined in equation (5) in the text. Inserting \(dV(W,t)\) into (A1), making use of

\[
E(dW) = \left[ (\mu_s - \mu_p - \sigma_{sp}^2 + \sigma_p^2) \right] dt - h \mu_p dt W - C dt
\]

\[
E(dW)^2 = \left[ (\sigma_s^2 dC_s - \sigma_p \sigma_p dC_p - h \sigma_p dC_p) \right] W^2 + o(dt)
\]

\[
= \left[ \sigma_s^2 + \sigma_p^2 - 2 \sigma_{sp} - 2h(\sigma_{sp} - \sigma_{pf}) + h^2 \sigma_p^2 \right] W^2 dt
\]

gives

\[
\beta V(W,t) = \max_{C,h} \left[ U(C) + V_t + V_w \left[ (\mu_s - \mu_p - \sigma_{sp}^2 + \sigma_p^2) - h \mu_p \right] W - C \right]
\]

(A2)

which is equation (7) in the text. Maximization of the right hand side gives rise to the two first order conditions

\[
U_C(C) - V_w(W,t) = 0
\]

(A3.1)

\[
- \mu_p V_w W + \frac{1}{2} V_{ww} \left[ (-2)(\sigma_{sp} - \sigma_{pf}) + 2h \sigma_p^2 \right] W^2 = 0
\]

(A3.2)

from which (8a) and (8b) immediately follow.

2. Optimality conditions in case of the isoelastic utility function and infinite planning horizon

\[
U(C) = \frac{1}{\gamma} C^\gamma, \quad -\infty < \gamma < 1
\]

Inserting this utility function and the optimized values \(\hat{C}\) and \(\hat{h}\) into the Bellman equation (A2), writing \(V(W)\) instead of \(V(W,t)\), ignoring thus the \(V_t\)-term on the right hand side, and rearranging yields
\[
\begin{align*}
\left\{ \frac{1}{\gamma} \hat{C}' + \left[ \hat{a} - \frac{\hat{C}}{W} \right] WV_W(W) + \frac{1}{2} \hat{b} W^2 V_{WW}(W) \right\} - \beta V(W) &= 0 \\
(A3)
\end{align*}
\]

with \( \hat{a} \) and \( \hat{b} \) defined as in the text. This is an ordinary differential equation. Note that since we have inserted optimized values, the “max”-operator on the right hand side of (A2) drops out. The solution of the differential equation (A3) is by trial and error. Looking at (A3), it is quite natural to postulate a solution of the form (see Turnovsky (2000), ch. 15)

\[
V(W) = \delta W^\gamma
\]

where the coefficient \( \delta \) has to be determined. (A4) implies

\[
V_w = \delta \gamma W^\gamma - 1, \quad V_{ww} = \delta \gamma (\gamma - 1) W^{\gamma - 1}
\]

Substituting (A4) and (A5) into the first order conditions (A3.1) and (A3.2), using \( U_c = C^{\gamma - 1} \) yields

\[
\dot{C}^{\gamma - 1} = \delta \gamma W^{\gamma - 1}, \quad \text{or} \quad \frac{\dot{C}}{W} = (\delta \gamma)^{1 - \gamma - 1}
\]

\[
\begin{align*}
- \mu \delta \gamma W^{\gamma} + \delta \gamma (\gamma - 1) W^{\gamma} \left[ (-1)(\sigma_{sf} - \sigma_{pf}) + \hat{h}\sigma_s^2 \right] &= 0, \quad \text{or equivalently}
\end{align*}
\]

\[
\dot{h} = \frac{\sigma_{sf} - \sigma_{pf}}{\sigma_s^2} + \frac{\mu \gamma}{\sigma_s^2 (\gamma - 1)}
\]

(A6.2)

which is eq. (9c) in the text.

3. Solution of the Bellman equation

Next, we solve (A6.1) for \( \dot{C} = (\delta \gamma)^{1 - \gamma - 1} W \) and insert this together with (A4) and (A5) for \( V(W), V_w, V_{ww} \) respectively, into the Bellman differential equation (A3) to get

\[
\frac{1}{\gamma} (\delta \gamma)^{1 - \gamma - 1} + \left[ \hat{a} - (\delta \gamma)^{1 - \gamma - 1} \right] \delta \gamma + \frac{1}{2} \hat{b} \delta \gamma (\gamma - 1) - \beta \delta = 0
\]

This equation can be solved for

\[
(\delta \gamma)^{1 - \gamma - 1} = \frac{\beta - \dot{a} \gamma (1/2) \hat{b} \gamma (\gamma - 1)}{1 - \gamma}
\]

Combining this expression with (A6.1), we get the consumption-wealth-ratio

\[
\frac{\dot{C}}{W} = \frac{\beta - \dot{a} \gamma (1/2) \hat{b} \gamma (\gamma - 1)}{1 - \gamma}
\]

(A7)

This is eq. (9b) in the text. Solving (A6.1) for the coefficient \( \delta \) gives
\[ \delta = \frac{1}{\gamma} \left( \hat{C}/W \right)^{\gamma^{-1}} \]  

(A8)

where \( \hat{C}/W \) is given by (A7). Thus, the value function becomes

\[ V(W) = \frac{1}{\gamma} \left( \hat{C}/W \right)^{\gamma^{-1}} W^\gamma \]  

or

\[ V(W) = \frac{W^\gamma}{\gamma \left( \hat{C}/W \right)} \]

(A9)

where the first equation is (9a) in the text. Inserting (A7) into (A9), assuming \( \hat{h} \neq 0 \), and using the initial level of real wealth \( W(0) \), in case of \( 0 < \gamma < 1 \) we get

\[ (\hat{C}/W(0)) < (\hat{C}/W(0)) \text{ without hedging} \]  

and thus

\[ V(W(0)) > V(W(0)) \text{ without hedging} \],  

and for

\[ -\infty < \gamma < 0 \quad (\hat{C}/W(0)) > (\hat{C}/W(0)) \text{ without hedging} \]  

and so again

\[ V(W(0)) > V(W(0)) \text{ without hedging} \]

(note that in this case the denominator of (A9) is negative because of \( \gamma < 0 \)).

In the case of the logarithmic utility function, \( \gamma = 0 \), and thus \( U(C) = \ln C \). A solution of the Bellman differential equation (A3) where \( \hat{C}^\gamma / \gamma \) is substituted by \( \ln C \), is

\[ V(W) = \lambda + \delta \ln W \].

From (A7) we then get \( \hat{C}/W = \beta \), both with and without hedging, since the optimal hedging strategy, enclosed in \( \hat{a} \) and \( \hat{b} \), does not influence the optimal consumption-wealth-ratio (A7). However, it can be shown that

\[ \lambda = \frac{\beta \ln \beta - \beta + \hat{a} - \hat{b}/2}{\beta^2}, \quad \delta = \frac{1}{\beta} \left( \hat{C}/W \right)^{-1} \]

and that

\[ \hat{a} - \frac{1}{2} \hat{b} = (\mu_S - \mu_P - \sigma_{SP} + \sigma_P^2) - \frac{1}{2} (\sigma_S^2 + \sigma_P^2 - 2\sigma_{SP}) + \frac{1}{2} \sigma_S^2 \hat{h}^2 \]

where the first two terms in parentheses denote \( a - b/2 \) in case without hedging. Hence

\[ \lambda > \lambda \text{ without hedging} \]  

thus proving

\[ V(W(0)) > V(W(0)) \text{ without hedging} \].

From these observations inequality (10) in the text follows.
REFERENCES


Elder, J. (2004), Another Perspective on the Effects of Inflation Uncertainty, Journal of Money, Credit and Banking, 36, 911 – 928

Holthausen, D. M (1979), Hedging and the Competitive Firm under Price Uncertainty, American Economic Review, 69, 989 – 995


07/03 Karmann, Alexander / Maltritz, Dominik: Sovereign Risk in a Structural Approach

08/03 Friedrich, B. Cornelia: Internet-Ökonomie. Ökonomische Konsequenzen der Informations- und Kommunikationstechnologien (IuK)

09/03 Lehmann-Waffenschmidt, Marco: A Fresh Look on Economic Evolution from the Kinetic Viewpoint

10/03 Berlemann, Michael: The Effect of Signalling and Beliefs on the Voluntary Provision of Public Goods - Some Experimental Evidence

11/03 Berlemann, Michael / Nenovsky, Nikolay: Lending of First Versus Lending of Last Resort - The Bulgarian Financial Crisis of 1996/1997

12/03 Wälde, Klaus: Endogenous business cycles and growth

13/03 Choi, Jay Pil / Thum, Marcel: The economics of repeated extortion

14/03 Broll, Udo / Eckwert, Bernhard: Transparency in the Foreign Exchange Market and the Volume of International Trade


16/03 Steinmann, Lukas / Dittrich, Gunnar / Karmann, Alexander / Zweifel, Peter: Measuring and Comparing the (In)Efficiency of German and Swiss Hospitals

17/03 Lehmann-Waffenschmidt, Marco / Reina, Livia: Coalition formation in multilateral negotiations with a potential for logrolling: an experimental analysis of negotiators’ cognition processes

18/03 Lehmann-Waffenschmidt, Marco / Böhmer, Robert: Mentality Matters – Thorstein Veblens ’Regime of Status’ und Max Webers ’Protestantische Ethik’ aus der Sicht des (radikalen) Konstruktivismus. Eine Anwendung auf die ökonomischen Probleme des deutschen Wiedervereinigungsprozesses

19/03 Eisenschmidt, Jens / Wälde, Klaus: International Trade, Hedging and the Demand for Forward Contracts

20/03 Broll, Udo / Wong, Kit Pong: Capital Structure and the Firm under Uncertainty

01/04 Lehmann-Waffenschmidt, Marco: A Note on Continuously Decomposed Evolving Exchange Economies

02/04 Friedrich, B. Cornelia: Competition and the Evolution of Market Structure in the E-conomy.

03/04 Berlemann, Michael / Dittrich, Marcus / Markwardt, Gunther: The Value of Non-Binding Announcements in Public Goods Experiments. Some Theory and Experimental Evidence

04/04 Blum, Ulrich / Schaller, Armin / Veltins, Michael: The East German Cement Cartel: An Inquiry into Comparable Markets, Industry Structure, and Antitrust Policy

05/04 Schlegel, Christoph: Analytical and Numerical Solution of a Poisson RBC model

06/04 Lehmann-Waffenschmidt, Marco: Die ökonomische Botschaft in Goethes „Faust“

07/04 Fuchs, Michaela / Thum, Marcel: EU Enlargement: Challenges for Germany’s New Laender

08/04 Seitz, Helmut: Implikationen der demographischen Veränderungen für die öffentlichen Haushalte und Verwaltungen

09/04 Sülzle, Kai: Duopolistic Competition between Independent and Collaborative Business-to-Business Marketplaces

10/04 Broll, Udo / Eckwert, Bernhard: Transparency in the Interbank Market and the Volume of Bank Intermediated Loans

11/04 Thum, Marcel: Korruption

12/04 Broll, Udo / Hansen, Sabine / Marjit, Sugata: Domestic labor, foreign capital and national welfare

13/04 Nyamtseren, Lhamsuren: Challenges and Opportunities of Small Countries for Integration into the Global Economy, as a Case of Mongolia

01/05 Schubert, Stefan / Broll, Udo: Dynamic Hedging of Real Wealth Risk