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ASTRONOMICAL REFERENCE FRAMES IN THE PPN FORMALISM

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ABSTRACT. The necessity of the theory of local reference systems in the PPN formalism for adequate relativistic modeling of modern high-precision observations is discussed. The first results of the PPN theory of the local reference systems are described. A new effect in the rotational equations of motion of extended bodies in the PPN formalism resulting in the dependence of the rotational motion of a body on its acceleration relative to the barycentric reference system is discussed.

1. MOTIVATION

Relativistic effects play nowadays a quite important role in treating modern precise astronomical observations of many kinds. Appreciating this fact, International Astronomical Union recommended in 1991 to use some basic ideas of Einstein's General Relativity to model various high-precision observations (Bergeron, 1991).

Although General Relativity is consistent with all kinds of available observations, this theory is by no means the only possible relativistic theory of gravitation and the use of only this theory for modeling high-precision observation would contradict basic scientific principles. Moreover, modern astrometric and geodynamical observations are one of the most important sources of observational data allowing us to test General Relativity and other relativistic gravity theories. For this reason the International Earth Rotation Service (IERS) in the IERS Standards and Conventions (see, e.g. McCarthy, 1996) recommends to use relativistic models with two numerical parameters γ and β, characterizing possible deviations of General Relativity from physical reality. These IERS models are now used for many aspects of testing of General Relativity. For example, the current best estimates of γ and β come from processing of LLR and geodetic VLBI observations. On the other hand, the numerical values of many parameters (e.g., masses) determined from observations may be biased by wrong theoretical models and adding additional reasonable degrees of freedom coming from γ and β allows one to make systematic error analysis more justified.

The reason for the discrepancy between the purely Einsteinian recommendations of the IAU and the more general models of the IERS is the lack of theoretical foundation for the IAU recommendations in the framework of a broader class of relativistic gravity theories. Indeed, the recommendations of the IAU are based on the construction of the so-called local geocentric reference system, but rigorous theory of these local reference systems is developed only in General Relativity. It does not mean automatically that the models recommended by the IERS are wrong. However, the use of inadequate or unjustified relativistic models may lead to biased estimates of the parameters γ and β which have fundamental physical significance. It is clear that the situation should be changed and the corresponding theory of the local reference systems should be elaborated for a broader class of the theories of gravitation.

2. PRINCIPAL PROBLEM

One of the principle theoretical problems in relativistic modeling of observations is the construction of physically adequate local reference systems for each spatially bounded material subsystem taking part in the observed physical phenomena. Various parameters of the models defined in such local reference systems can be usually reasonably interpreted at some reasonable level of accuracy neglecting the fact that the parameters are defined in one particular reference system and have, generally speaking, no physical meaning. It is natural to suppose that a local reference system of a massive extended body, being a member of N massive extended bodies, has the following two properties:

(**A**) the gravitational field of external bodies is represented in the form of tidal potentials being $\mathcal{O}(\mathbf{X}^2)$, where X^i are local coordinates;

(**B**) the internal gravitational field of the body coincides with the gravitational field of a corresponding isolated source provided that the tidal influence of the external matter is neglected.

The principal physical idea of the local reference systems are illustrated in Figure 1. In present time there exist two well-developed theoretical formalisms to solve the problem of the local reference systems in General Relativity: the Brumberg-Kopeikin formalism (see, e.g., Brumberg, 1991; Kopeikin, 1991) and the DSX formalism (Damour, Soffel, Xu, 1991, 1992, 1993). The results of these formalisms prove that in General Relativity both properties **A** and **B** can be satisfied simultaneously. The IAU recommendations are based, in fact, on the results of these two formalisms. The aim of our work is to generalize principal ideas of the formalisms onto a broader class of relativistic theories of gravitation and in particular onto the PPN formalism.

3. WHAT IS THE PPN FORMALISM?

The parametrized post-Newtonian (PPN) formalism is a general phenomenological scheme covering a broad class of alternative relativistic theories of gravitation in the first post-Newtonian approximation. The PPN formalism contains a number of parameters having some particular numerical values in particular theories of gravitation. The idea of the PPN formalism is to write down the metric tensor of the barycentric reference system in the first post-Newtonian approximation with all kinds of gravitational potentials appearing within a class of gravity theories and to introduce a number of parameters as factors of these potentials. The simplest version of the PPN formalism results in the metric tensor of the barycentric reference system (t, x^i) with only two gravitational potentials w and w^i and two parameters γ and β

$$
\begin{aligned}
g_{00} &= 1 - \frac{2}{c^2}\, w(t,\mathbf{x}) + \frac{2}{c^4}\,\beta\, w^2(t,\mathbf{x}) + \mathcal{O}(c^{-5}),\\
g_{0i} &= \frac{2(1+\gamma)}{c^3}\, w^i(t,\mathbf{x}) + \mathcal{O}(c^{-5}),\\
g_{ij} &= -\delta_{ij}\left(1 + \frac{2}{c^2}\,\gamma\, U(t,\mathbf{x})\right) + \mathcal{O}(c^{-4}),
\end{aligned}
\qquad (1)
$$

where

$$
w = G\int \sigma(t,\mathbf{x}')\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,dx'^3 + \frac{1}{c^2}\,G\,\frac{\partial^2}{\partial t^2}\int \sigma(t,\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|\,dx'^3 + \mathcal{O}(c^{-4}), \qquad (2)
$$

$$
w^i = G\int \sigma^i(t,\mathbf{x}')\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,dx'^3 + \mathcal{O}(c^{-2}), \qquad (3)
$$

$$
\sigma = \frac{1}{c^2}\left(T^{00} + \gamma T^{kk} + \frac{1}{c^2}T^{00}\,(3\gamma - 2\beta - 1)w\right) + \mathcal{O}(c^{-4}), \qquad \sigma^i = \frac{1}{c}T^{0i} + \mathcal{O}(c^{-2}) \qquad (4)
$$

and $T^{\alpha\beta}$ is the energy-momentum tensor in the barycentric reference system. In Einstein's General Relativity $\gamma = \beta = 1$. This simplest version of the PPN formalism covers, nevertheless, the most viable

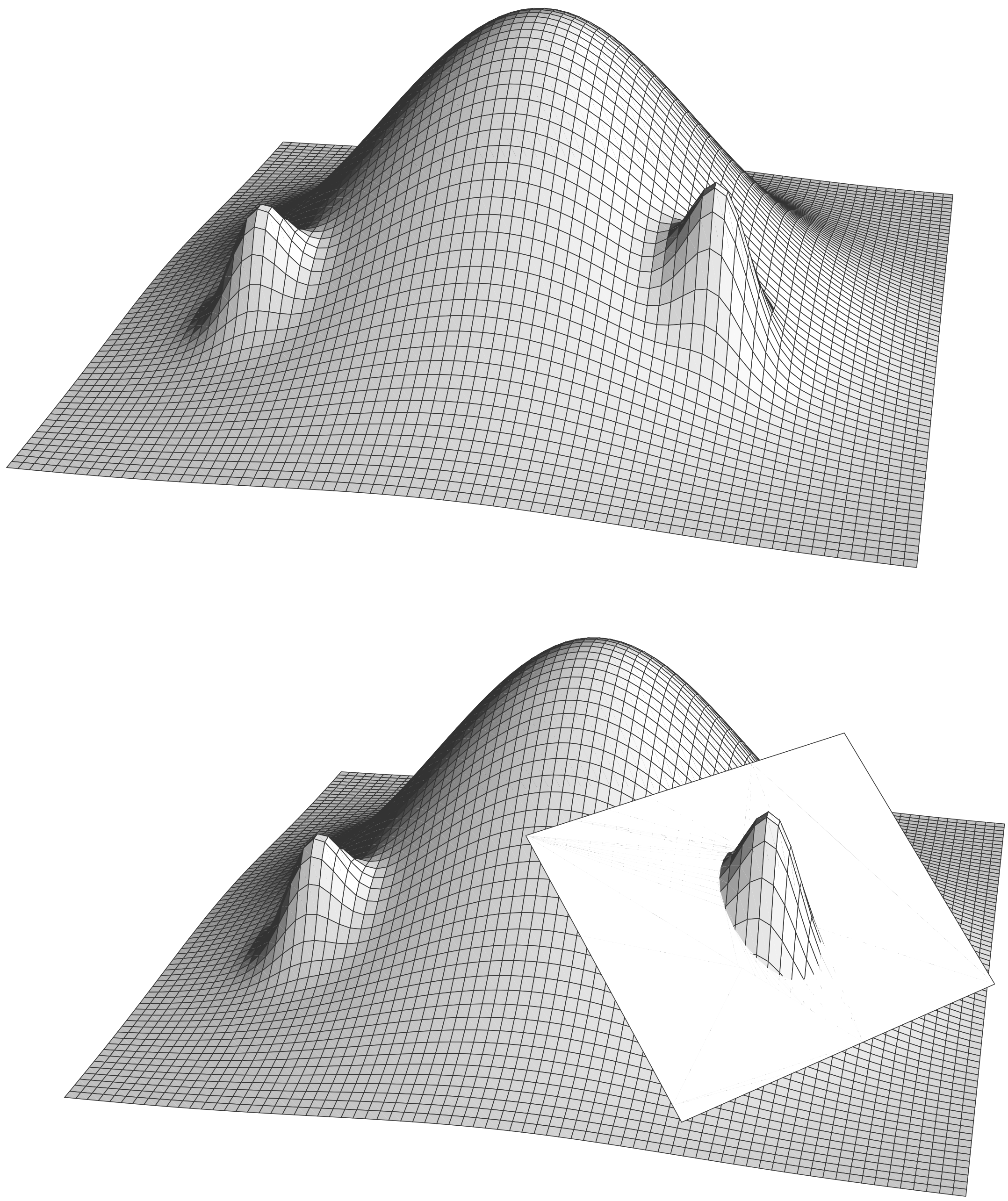

Figure 1: Basic idea of the local reference systems can be illustrated as follows. Upper picture is a graphical representation of the curved space-time produced by a system of one massive central body (say, the Sun) and two less massive bodies (say, two planets). If we describe the system as a whole we have to consider all three sources of curvature (that is, of gravitational field) uniformly. However, if we consider only a relatively small vicinity of one of the bodies, we note that the background curvature due to the two other bodies changes very slowly within that region. We, therefore, can introduce a local reference system (see, lower picture) in which we see effectively only the curvature due to the considered body. It is intuitively clear that the background curvature of the space-time still appear in the local reference system, but in the tidal form.

theories of gravitation. There exist also several more complicated versions of the PPN formalism covering a broader class of the theories and containing up tp 10 parameters. Here we confine ourselves by the simplest version (1)–(4). However, the method we use below to solve the problem can be applied to any version of the PPN formalism.

4. HOW TO SOLVE THE PROBLEM

The metric tensor $g_{\alpha\beta}$ of the barycentric reference system is defined by (1)–(4). Potentials w and w^i are defined as volume integrals over the whole space. Using this fact, we split the area of integration into the volume V of the body, for which we want to construct a local reference system, and the rest of the space. Thus, we split w and w^i into internal potentials (potentials of the body under consideration) and external ones (potentials due to the other (external) bodies):

$$w(t,\mathbf{x}) = w_E(t,\mathbf{x}) + \overline{w}(t,\mathbf{x}),$$
$$w^i(t,\mathbf{x}) = w^i_E(t,\mathbf{x}) + \overline{w}^i(t,\mathbf{x}). \tag{5}$$

We *suppose* apriori that, the metric tensor $G_{\alpha\beta}$ of the local reference system (T, X^i) has the same functional form as $g_{\alpha\beta}$

$$G_{00} = 1 - \frac{2}{c^2}\, W(T,\mathbf{X}) + \frac{2}{c^4}\,\beta\, W^2(T,\mathbf{W}) + \mathcal{O}(c^{-5}),$$
$$G_{0i} = \frac{2(1+\gamma)}{c^3}\, W^i(T,\mathbf{X}) + \mathcal{O}(c^{-5}),$$
$$G_{ij} = -\delta_{ij}\left(1 + \frac{2}{c^2}\,\gamma\, W(T,\mathbf{X})\right) + \mathcal{O}(c^{-4}), \tag{6}$$

and the local gravitational potentials W and W^i can be represented in the form

$$W(T,\mathbf{X}) = W_E(T,\mathbf{X}) + Q_i(T)\, X^i + W_T(T,\mathbf{X}) + \frac{1}{c^2}\Psi(T,\mathbf{X}), \tag{7}$$

$$W^i(T,\mathbf{X}) = W^i_E(T,\mathbf{X}) + \frac{1}{2}\,\varepsilon_{ijk}\, C_j(T)\, X^k + W^i_T(T,\mathbf{X}). \tag{8}$$

Here W_E and W^i_E are defined by the same kind of integrals over the volume of the body as W_E and W^i_E, but all the quantities should be taken in the local reference system. For example, the energy-momentum tensor in (4) should also be computed in the local reference system (T, X^i). It is also *supposed* that the external potentials W_T and W^i_T are tidal, that is, of order of $\mathcal{O}(\mathbf{X}^2)$. $Q_i(T)$ and $C_i(T)$ characterize the world line of the origin of the local reference system and the spatial orientation of the local reference system relative to the barycentric one. Ψ is some unknown function to be derived by a later check of consistency of all the adopted assumption (see below). From physical considerations one can guess that if the Equivalence Principle (on which General Relativity is heavily based) is violated, it is impossible to satisfy both properties **A** and **B** simultaneously. Accounting for this fact we suppose that W contains an unknown function Ψ, which results from the possible violation of property **B**. On the other hand, by requiring that the external potentials W_T and W^i_T are tidal ($\sim \mathcal{O}(\mathbf{X}^2)$) we apriori *assume* that property **A** is satisfied.

Coordinate transformations between the local and barycentric reference system are *supposed* to have the form

$$T = t - \frac{1}{c^2}\left(A + \dot{x}^i_E r^i_E\right) + \frac{1}{c^4}\left(B + B^k r^k_E + B^{km} r^k_E r^m_E + C(t,\mathbf{x})\right) + \mathcal{O}(c^{-5}), \tag{9}$$

$$X^i = R^i_{\ j}\left(r^j_E + \frac{1}{c^2}\left[\left(\frac{1}{2}\dot{x}^j_E\dot{x}^k_E + D^{jk}\right) r^k_E + D^{jkl} r^k_E r^l_E\right]\right) + \mathcal{O}(c^{-4}), \tag{10}$$

where $A(t)$, $B(t)$, $B^k(t)$, $B^{km}(t) = B^{mk}(t)$, $C(t,\mathbf{x}) \sim \mathcal{O}(|\mathbf{r}_E|^3)$, orthogonal matrix $R^i_{\ j}(t)$, $D^{ij}(t) = D^{ji}(t)$, $D^{ijk}(t) = D^{ikj}(t)$ are some unknown functions, $r^i_E = x^i - x^i_E(t)$, and $x^i_E(t)$ are the barycentric coordinates of the origin of the local reference system.

The unknown functions from the coordinate transformations A, B, B^k, B^{km}, C, $R^i_{\ j}$, D^{ij}, D^{ijk} and x^i_E, as well as those from the metric tensor $G_{\alpha\beta}$ of the local reference system W_T, W^i_T, C_i, Q_i and Ψ are derived or constrained from matching of the metric tensors of the local and barycentric reference systems

$$g_{\alpha\lambda}(t,\mathbf{x}) = \frac{\partial X^\mu}{\partial x^\alpha}\,\frac{\partial X^\nu}{\partial x^\lambda}\, G_{\mu\nu}(T,\mathbf{X}), \tag{11}$$

which can be considered as simply a consistency check of all the assumptions on the structure of the local metric tensor (6)–(8), the coordinate transformations (9)–(10) as well as on the properties of various functions mentioned above.

5. SOME RESULTS

Matching (11) enables one to derive all the unknown functions and complete therewith the construction of the local metric tensor and the coordinate transformations between the local and barycentric PPN reference systems. As we see both the metric and the transformations can be derived in closed form without expansions in powers of local coordinates X^i. Detailed results of the matching will be published elsewhere. However, one result we show here. Matching shows that

$$\Psi = -\eta \, \left(w_E(t,\mathbf{x}) \left(\overline{w}(t,\mathbf{x}_E(t)) + \ddot{x}^k_E r^k_E \right) - \chi^E_{,k} \ddot{x}^k_E(t) \right) + \mathcal{O}(c^{-2}), \tag{12}$$

$$\chi_E(t,\mathbf{x}) = \frac{1}{2} G \int_V \sigma(t,\mathbf{x}') \, |\mathbf{x}-\mathbf{x}'| \, dx'^3 + \mathcal{O}(c^{-2}). \tag{13}$$

This means that property **B** is violated provided that the Nordtvedt parameter $\eta = 4\beta - \gamma - 3$, reflecting violation of the Equivalence Principle, does not vanish. It is easy to prove also that one can construct another version of the local reference system which satisfies property **B**, but violates property **A**. However, in case $\eta \neq 0$ it is impossible to construct a local reference system satisfying both properties simultaneously.

Using the metric tensor of the local reference system and geodetic equations $d^2X^\mu/d\lambda^2 + \Gamma^\mu_{\alpha\nu}\, dX^\alpha/d\lambda \times$ $\times dX^\nu/d\lambda = 0$, one can derive equations of light propagation and translational equations of motion of a massless observer relative to the local reference system. From the local equations of motion $T^{\mu\nu}_{\ \ ;\nu} = 0$ written in the local reference system one gets translational equations of motion of N extended bodies as well as rotational equations of motion of each body relative to its local reference system. Details of these equations will be published elsewhere. All these equations together with the corresponding theory of observables based again on the metric tensors of both reference systems allows one to construct PPN models of any kind of astronomical observations.

For practical use of the equations of motion one usually prefers to work with multiple expansions of the equations. A physically adequate definition of multipole moments is very important here. In general-relativistic celestial mechanics the so-called Blanchet-Damour multipole moments are known to play this fundamental role. A slightly different definition of multipole moments intended to play the same role in the PPN celestial mechanics are given in Klioner, Soffel (1998).

6. ROTATIONAL EQUATIONS OF MOTION

The local reference system of an extended massive body described above allows us to derive the rotational equations of motion of the body in the framework of the PPN formalism. From the equation ($\mathcal{G}$ is the determinant of the local metric tensor and $\mathcal{T}^{\alpha\beta}$ is the energy-momentum tensor in the local reference system)

$$\varepsilon_{ijk} \int_V (-\mathcal{G}) \, X^j \, \mathcal{T}^{k\beta}_{\ \ ;\beta} dX^3 = 0, \tag{14}$$

which is valid due to the local equations of motion

$$\mathcal{T}^{\alpha\beta}_{\ \ \ ;\beta} = 0, \tag{15}$$

in analogy to General Relativity (see, Damour, Soffel, Xu, 1993; Klioner, 1996) one can derive the rotational equations of motion for the body

$$\frac{d}{dT} \, S^i = L^i_{\rm tidal} + L^i_{\rm Nor} + \mathcal{O}(c^{-4}), \tag{16}$$

where the PPN spin S^i is defined by

$$S^i = \varepsilon_{ijk} \int_V X^j \, p^k(T,\mathbf{X}) \, dX^3 + \mathcal{O}(c^{-4}), \tag{17}$$

$$p^i = \Sigma^i \,(1 + \frac{5\gamma - 1}{c^2}\, W) - \frac{1}{2c^2}\, G\, \Sigma \int_V \Sigma^j(T, \mathbf{X}')\; \frac{(4\gamma+3)\,\delta^{ij} + n^i n^j}{|\mathbf{X} - \mathbf{X}'|}\; dX'^3 + \mathcal{O}(c^{-4}), \tag{18}$$

$$n^i = \frac{X^i - X'^i}{|\mathbf{X} - \mathbf{X}'|}, \tag{19}$$

$$\Sigma = \frac{1}{c^2}\left(\mathcal{T}^{00} + \gamma \mathcal{T}^{kk} + \frac{1}{c^2}\mathcal{T}^{00}\,(3\gamma - 2\beta - 1)\, W\right) + \mathcal{O}(c^{-4}), \qquad \Sigma^i = \frac{1}{c}\mathcal{T}^{0i} + \mathcal{O}(c^{-2}), \tag{20}$$

the PPN torque $L^i_{\rm tidal}$ reads

$$L^i_{\rm tidal} = \varepsilon_{ijk} \int_V X^j\, f^k\, dX^3 + \mathcal{O}(c^{-4}), \tag{21}$$

$$f^i = \Sigma\, Q^i + \Sigma\, W_{T,i} + \frac{1}{c^2}\left\{ 2(1+\gamma)\,\left(\Sigma\, \frac{\partial}{\partial T} W^i_T + \Sigma^j\,\left(W^i_{T,j} - W^j_{T,i}\right)\right) + (1+\gamma)\, \varepsilon_{ijk}\,\left(2C_j\, \Sigma^k + \Sigma\, \dot{C}_j\, X^k\right)\right\}, \tag{22}$$

and the additional torque $L^i_{\rm Nor}$ is defined as

$$L^i_{\rm Nor} = \frac{1}{c^2}\, \eta\, \varepsilon_{ijk} \int_V \Sigma\, X^j\, \Psi_{,k}\, dX^3 = \frac{1}{c^2}\, \eta\, \varepsilon_{ijk}\, \Omega^j_E\, \ddot{x}^k_E + \mathcal{O}(c^{-2}), \tag{23}$$

$$\Omega^i_E = -\frac{1}{2} \int_V \Sigma\;\, W_E\, X^i\, dX^3 + \mathcal{O}(c^{-2}). \tag{24}$$

The PPN tidal torque $L^i_{\rm tidal}$, as in General Relativity, vanishes for case of an isolated body.The additional torque $L^i_{\rm Nor}$ represents an analogy of the so-called Nordtvedt effect in the rotational equations of motion. $L^i_{\rm Nor}$ is proportional to the Nordtvedt parameter $\eta = 4\beta - \gamma - 3$, which is not zero in a particular theory of gravitation only if that theory leads to a violation of the Equivalence Principle. This additional torque in the rotational equations of motion enables us to formulate a new test of General Relativity: if $4\beta - \gamma - 3 \neq 0$ the rotational motion of a body depends on its acceleration relative to the barycentric reference system. This effect is a direct consequence of the violation of the Equivalence Principle. The torque $L^i_{\rm Nor}$ is proportional also to the integral Ω^i_E which is obviously vanishes for a spherically symmetric body. Roughly speaking, for Ω^i_E to be non-zero the body should be non-symmetric with respect to its center of mass. Detailed estimates of the effect for real celestial bodies (first of all, for the Earth and the Moon) are still to be done. Nevertheless, in principle the effect in question allows one to check the validity of the Equivalence Principle with the use of another kind of observations: observations of the rotational motion of celestial bodies.

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