On the Expansions of Intermediate Orbit for General Planetary Theory

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Abstract. Intermediate orbit for general planetary theory is constructed in the form of multivariate Fourier series with numerical coefficients. The structure and efficiency of the derived series are illustrated by giving various statistical properties of the coefficients.

The ability of the recently proposed elliptic function approach to compress the Fourier series representing the intermediate orbit is investigated. Our results confirm that when mutual perturbations of a pair of planets are considered the elliptic function approach is quite efficient and allows one to compress the series substantially. However, when perturbations of three or more planets are under study the elliptic function approach does not give any advantages.

Key words: general planetary theory, elliptic functions

1. Introduction

Constructing general planetary theory (GPT), that is an analytical theory of planetary motion in a purely trigonometric form, which is formally valid for any moment of time, was considered as a central problem of celestial mechanics. Already by the beginning of this century it has been proved that the planetary motion could be formally represented in a purely trigonometric form. Several algorithms to construct the GPT have also been proposed at that time. However, technical difficulties did not allow one to construct the theory in practice (see, e.g., Brumberg, 1966, 1970 for review). In 60s a new algorithm has been proposed based on the Hill's lunar method (Brumberg, 1970). During next decade several steps to construct [some parts of] the GPT in practice have been undertaken by the research groups in the Institute of Theoretical Astronomy (Leningrad) and Bureau des Longitudes (Paris). The first-order theory, the intermediate orbit and the linear second-order theory have been discussed in (Brumberg, Chapront, 1973; Brumberg, Evdokimova, Skripnichenko, 1975, 1978). However, it has become clear that even the use of computers with special computer algebra software did not allow at that time to overcome technical difficulties: the resulting series were too voluminous if a reasonable accuracy was considered.

Since that time performance of computers has become several orders of magnitude higher and continues to grow quickly. We believe that the

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GPT in the form proposed in (Brumberg, 1970) could be constructed now with some reasonable accuracy, at least, in a semi-analytical form of Fourier series with numerical coefficients. However, it is quite clear that, besides the problems of convergence typical for any analytical theory of motion, we will be faced the problem to operate with extremely lengthy series. One of the possible remedies is to find a way to represent the resulting analytical theory in a more compact form.

In 80s several authors suggested to use elliptic functions and integrals in planetary theories (Richardson, 1982; Williams, Van Flandern, Wright, 1987; Williams, 1992). The use of elliptic functions allowed one to avoid the expansions of $1/\delta_{ij}$, where

$$\delta_{ij} = \left(a_i^2 + a_j^2 - 2a_i a_j \cos(\lambda_i - \lambda_j)\right)^{1/2} \tag{1}$$

is the distance between the ith and jth planets on coplanar circular orbits, a_i is the semi-major axis of the corresponding planet and λ_i is its mean longitude. The expansion of $1/\delta_{ij}$ in powers of the ratio $\alpha_{ij} = \min(a_i/a_j, a_j/a_i)$ is known to converge very slowly when α_{ij} is close to 1 and presents one of the most serious practical difficulties when constructing a [semi-]analytical planetary theory. The principal idea is to introduce new angular variables u_{ij}

$$u_{ij} = F(\varphi_{ij}, k_{ij}), \tag{2}$$

$$k_{ij}^2 = \frac{4a_i a_j}{(a_i + a_j)^2} = \frac{4\alpha_{ij}}{(1+\alpha_{ij})^2}, \tag{3}$$

instead of the usual synodic variables

$$\varphi_{ij} = \frac{1}{2}\left(\pi - (\lambda_i - \lambda_j)\right) \tag{4}$$

Here $F(\varphi, k)$ is the elliptic integral of the first kind. Effectively the new angular variable (2) is a change of time variable. Using these variables one could represent the distance between planets in closed form in terms of elliptic functions: $\delta_{ij} = (a_i + a_j)\,\mathrm{dn}(u_{ij}, k_{ij})$. Williams, Van Flandern and Wright (1987) have shown that the first-order perturbations can be also represented in closed form in terms of elliptic functions and elliptic integrals.

However, an extension of these results onto higher orders seems to be difficult (Brumberg, 1992), since it would require operating with expressions which involve elliptic functions and integrals with several different arguments. For example, it would be necessary to take in symbolic form the integrals like $\int \mathrm{cn}(u_{ij}, k_{ij})\mathrm{cn}(u_{ik}, k_{ik}) du_{ij}$. Chapront and Simon (1988), and independently Brumberg (1992) suggested to use the expansions of elliptic functions and integrals in their Fourier series. These expansions converge rather quickly even for large values of α_{ij} since the principal small parameter of the expansions is not α_{ij}, but the nome $q_{ij} = \exp\left(-\pi K(k'_{ij})/K(k_{ij})\right)$, where $K(k)$ is the complete elliptic integral of the first kind, $k' = \sqrt{1-k^2}$. The nome q is small enough even for large α. Indeed, for the major planets the largest value of α_{ij} is $\alpha_{23} = 0.723$ for the couple Venus–Earth. For the same couple $q_{23} = 0.215$. This allowed one to hope that the resulting series converge much faster than the classical developments (see, Brumberg, 1992, 1996). Chapront and Simon (1988, 1996) showed that the use of u_{ij} as angular variables leads to more compact series when the mutual perturbations of a couple of planets are considered in a classical-type planetary theory.

Brumberg (1992, 1994) suggested to apply the elliptic function approach to the GPT and re-constructed the first-order GPT, as developed in Brumberg and Chapront (1973), in the form of series with closed-form coefficients expressed in terms of elliptic functions. The first-order intermediary has been represented therewith as a closed-form expression involving elliptic functions and integrals, and Hansen-type quadratures with trigonometric kernels. The higher-order theory can also be constructed with the aid of Fourier expansions of elliptic functions and integrals. As far as the GPT is concerned, the elliptic function approach effectively consists in constructing the theory in the form of the series, which we will call τ-series (see, Brumberg, 1992):

$$\sum_{k_1,k_2,\ldots,k_n} \mathcal{B}_{k_1,k_2,\ldots,k_n} \tau_{i_1j_1}^{k_1} \tau_{i_2j_2}^{k_2} \ldots \tau_{i_nj_n}^{k_n}, \tag{5}$$

$$\tau_{ij} = \exp\left[\mathring{\imath}\, w_{ij}\right], \qquad w_{ij} = \frac{\pi}{K(k_{ij})} u_{ij}, \tag{6}$$

where i_s and j_s are indices numbering the planets, and $\mathring{\imath} = \sqrt{-1}$. The classical developments, which we will call ρ-series, are

$$\sum_{k_1,k_2,\ldots,k_n} \mathcal{A}_{k_1,k_2,\ldots,k_n} \rho_{i_1j_1}^{k_1} \rho_{i_2j_2}^{k_2} \ldots \rho_{i_nj_n}^{k_n}, \tag{7}$$

$$\rho_{ij} = \exp\left[\mathring{\imath}\, l_{ij}\right], \qquad l_{ij} = 2\varphi_{ij}. \tag{8}$$

Following these lines, several steps toward constructing the GPT in the form of the τ-series have been undertaken in Brumberg (1992), Klioner (1992), and Brumberg and Klioner (1996).

The principal aims of this work are to demonstrate practical feasibility of constructing both the ρ-series and the τ-ones for the intermediate orbit for the GPT as well as to compare numerical efficiency of the two kind of series. The numerical efficiency has been already investigated in Brumberg and Klioner (1996) for the first-order intermediary. The conclusions were rather promising: the τ-series were shown to be more compact than the corresponding ρ-series. In Section 2 we describe a numerical algorithm allowing one to compute both ρ- and τ-series for higher-order GPT. In Section 3 we extend the results of Brumberg and Klioner (1996) for mutual perturbations of a pair of planets up to the fourth order with respect to the ratio μ of the planetary masses to the mass of the central body. Section 4 is devoted to a comparison of the efficiency of the ρ-series and the τ-series representing mutual perturbations of a triplet of planets of the second and third orders with respect to μ. Principal conclusions and some additional remarks are given in Section 5.

2. Intermediate Orbit in Semi-Numerical Form

The algorithm for constructing the GPT as suggested in Brumberg (1970) introduces new complex variables p_i and real variables w_i

$$x_i + \mathring{\imath}\, y_i = a_i(1-p_i)\exp \mathring{\imath}\, \lambda_i, \quad z_i = a_i w_i \tag{9}$$

instead of the heliocentric rectangular coordinates x_i, y_i, z_i, $i = 1,2,\ldots,N$ of the N planets. These dimensionless variables represent small deviations from the plane circular motion with a semi-major axis a_i and a mean longitude λ_i. The equations of motion in these new variables read

$$\ddot{p}_i + 2\,\mathring{\imath}\, n_i\dot{p}_i - \frac{3}{2}n_i^2(p_i+\overline{p_i}) \;=\; n_i^2 P_i, \tag{10}$$

$$\ddot{w}_i + n_i^2 w_i \;=\; n_i^2 W_i, \tag{11}$$

where here and below for any x, $\overline{x}$ is its complex conjugate, n_i is the mean motion of the ith planet, and the right-hand members read

$$P_i = -1 - \frac{1}{2}p_i - \frac{3}{2}\overline{p_i} + \frac{2}{n_i^2 a_i^2}\frac{\partial U_i}{\partial \overline{p_i}}, \quad W_i = w_i + \frac{1}{n_i^2 a_i^2}\frac{\partial U_i}{\partial w_i}, \tag{12}$$

U_i being the force function for the equations in the coordinates x_i, y_i, z_i.

The intermediate orbit $p_i = p_i^{(0)}$, $w_i = 0$ is a partial quasi-periodic solution of (10)–(11) representing the mutual perturbations of planets placed on coplanar circular orbits. The intermediary $p_i^{(0)}$ must satisfy Eq. (10) with some right-hand side $P_i^{(0)}$. The solution of (10) is constructed by successive approximations in powers of the ratio μ of the planetary masses to the mass of the central body

$$p_i^{(0)} = \sum_{k=1}^{\infty} \mu^k \underset{k}{p}_i^{(0)}, \tag{13}$$

where $\underset{k}{p}_i^{(0)}$ is the kth-order terms with respect to μ which, in turn, can be split into pieces induced by the perturbations from different sets of planets

$$\begin{aligned} \underset{1}{p}_i^{(0)} &= \sum_{j=1}^{N}{}^{(i)} \underset{1}{T}_j^{(i)}, & (14)\\ \underset{2}{p}_i^{(0)} &= \sum_{j=1}^{N}{}^{(i)} \underset{2}{T}_j^{(i)} + \sum_{j=1}^{N}{}^{(i)} \sum_{k=j+1}^{N}{}^{(i)} \underset{2}{T}_{jk}^{(i)}, & (15)\\ &\ldots. \end{aligned}$$

Here N is the number of planets. The right-hand side $P_i^{(0)}$ for the intermediary can also be split in the same manner: into pieces of different orders with respect to μ and induced by perturbations from different sets of planets (see, Brumberg, 1992, 1994 for details)

$$P_i^{(0)} = \sum_{k=1}^{\infty} \mu^k \underset{k}{P}_i^{(0)}, \tag{16}$$

$$\begin{aligned} \underset{1}{P}_i^{(0)} &= \sum_{j=1}^{N}{}^{(i)} \underset{1}{Q}_j^{(i)}, & (17)\\ \underset{2}{P}_i^{(0)} &= \sum_{j=1}^{N}{}^{(i)} \underset{2}{Q}_j^{(i)} + \sum_{j=1}^{N}{}^{(i)} \sum_{k=j+1}^{N}{}^{(i)} \underset{2}{Q}_{jk}^{(i)}, & (18)\\ &\ldots\ , \end{aligned}$$

$$\begin{aligned} \underset{1}{Q}_j^{(i)} &= \kappa_{ij}\psi_{0000}^{(ij)}, & (19)\\ \underset{2}{Q}_j^{(i)} &= \frac{3}{8}\left(\underset{1}{T}_j^{(i)}\right)^2 + \frac{3}{4}\underset{1}{T}_j^{(i)}\overline{\underset{1}{T}_j^{(i)}} + \frac{15}{8}\left(\overline{\underset{1}{T}_j^{(i)}}\right)^2 \\ &\quad + \kappa_{ij}\left(\psi_{1000}^{(ij)}\underset{1}{T}_j^{(i)} + \psi_{0100}^{(ij)}\overline{\underset{1}{T}_j^{(i)}} + \psi_{0010}^{(ij)}\underset{1}{T}_i^{(j)} + \psi_{0001}^{(ij)}\overline{\underset{1}{T}_i^{(j)}}\right), & (20) \end{aligned}$$

$$\begin{aligned}
\underset{2}{Q}{}^{(i)}_{jk} &= \frac{3}{4}\underset{1}{T}{}^{(i)}_{j}\underset{1}{T}{}^{(i)}_{k} + \frac{3}{4}\underset{1}{T}{}^{(i)}_{j}\overline{\underset{1}{T}{}^{(i)}_{k}} + \frac{3}{4}\underset{1}{T}{}^{(i)}_{k}\overline{\underset{1}{T}{}^{(i)}_{j}} + \frac{15}{4}\overline{\underset{1}{T}{}^{(i)}_{j}}\,\overline{\underset{1}{T}{}^{(i)}_{k}} \\
&+\kappa_{ij}\left(\psi^{(ij)}_{1000}\underset{1}{T}{}^{(i)}_{k} + \psi^{(ij)}_{0100}\overline{\underset{1}{T}{}^{(i)}_{k}} + \psi^{(ij)}_{0010}\underset{1}{T}{}^{(j)}_{k} + \psi^{(ij)}_{0001}\overline{\underset{1}{T}{}^{(j)}_{k}}\right), \\
&+\kappa_{ik}\left(\psi^{(ik)}_{1000}\underset{1}{T}{}^{(i)}_{j} + \psi^{(ik)}_{0100}\overline{\underset{1}{T}{}^{(i)}_{j}} + \psi^{(ik)}_{0010}\underset{1}{T}{}^{(k)}_{j} + \psi^{(ik)}_{0001}\overline{\underset{1}{T}{}^{(k)}_{j}}\right) \qquad (21)
\end{aligned}$$

$\dots$.

Here $\mu\kappa_{ij} = \dfrac{m_j}{m_0 + m_i}$, where m_i is the mass of the ith planet and m_0 is the mass of the central body,

$$\begin{aligned}
\psi^{(ij)}_{klrs} &= \frac{\left(\frac{1}{2}\right)_{k+r}\left(\frac{3}{2}\right)_{l+s}}{(1)_{k+r}\,(1)_{l+s}}\left[-\left(\frac{a_i}{a_j}\right)^2 \rho_{ij}\,\mathcal{S}(k,l) + \frac{(1+r)_k\,(1+s)_l}{(1)_k\,(1)_l}\times\right. \\
&\left.\times\gamma\left(r+s, -k-r-\frac{1}{2}, -l-s-\frac{3}{2}, -r+s, \frac{a_j}{a_i}, -\rho_{ij}^{-1}\right)\right], \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\gamma\left(n, -K/2, -L/2, \nu, \alpha, \xi\right) &= \alpha^n\,\frac{(-\xi)^\nu}{\left(1+\alpha^2-\alpha(\xi+\xi^{-1})\right)^{\max(K/2,L/2)}}\times \\
&\times\left(1-\alpha\,\xi^{\mathrm{sign}(K-L)}\right)^{|K-L|/2}, \qquad (23)
\end{aligned}$$

K and L are positive integers, $\mathcal{S}(k,l)=1$ if $k=l=0$ and $\mathcal{S}(k,l)=0$ otherwise.

Since Eq. (10) is linear, each piece of $p_i^{(0)}$ in (13)–(15) is generated by the corresponding piece of $P_i^{(0)}$ in (16)–(18). Therefore, the terms in (13)–(15) can be calculated subsequently. Eqs. (10) and (13)–(23) enable us to compute the intermediary to any desired order with respect to μ. We adopt the following procedure to compute the Fourier expansions of $p_i^{(0)}$ into the ρ- and the τ-series.

A. We compute the ρ-series of a piece of $P_i^{(0)}$ by numerical Fourier analysis (tabulating the corresponding terms in (19)–(21) and making use of the fast Fourier transform (FFT)). Hence, we get the expansions

$$\text{a piece of } P_i^{(0)} = \sum Q^i_{k_1k_2\dots k_n}\rho^{k_1}_{i_1j_1}\rho^{k_2}_{i_2j_2}\,\dots\,\rho^{k_n}_{i_nj_n} \qquad (24)$$

The number n of the angular variables $\rho_{i_sj_s}$ in (24) depends on the particular part of $P_i^{(0)}$ which we consider. For $\underset{k}{Q}{}^{(i)}_{j}$ the series (24) involve only one angular variable and $n=1$, while for $\underset{k}{Q}{}^{(i)}_{jk}$ $n=2$, etc. The sets i_s and j_s are not unique: for example, we can expand $\underset{k}{Q}{}^{(i)}_{jk}$ into double Fourier series in powers of ρ_{ij} and ρ_{ik}, in powers of ρ_{ij} and ρ_{jk}, or in powers of ρ_{ik} and ρ_{jk}, having respectively three different sets of i_s and j_s and three different forms of the series (24). However, taking into consideration the identities $\rho_{ji} = \rho_{ij}^{-1}$ and $\rho_{ij}\rho_{jk} + \rho_{ik} = 0$ we see that having computed the Fourier coefficients for one kind of the ρ-series one can immediately restore the coefficients of any other kind by relevant re-numbering of the coefficients. For example,

$$\sum_{a,b} \mathcal{A}_{a,b}\rho_{ij}^{a}\rho_{ik}^{b} = \sum_{a,b}(-1)^b \mathcal{A}_{a-b,b}\rho_{ij}^{a}\rho_{jk}^{b}. \tag{25}$$

Besides that, the ρ-series for the intermediary can be represented in the form

$$\begin{aligned} &\sum_{k_1,k_2,\ldots,k_n} \mathcal{A}_{k_1,k_2,\ldots,k_n}\rho_{i_1j_1}^{k_1}\rho_{i_2j_2}^{k_2}\ \ldots\ \rho_{i_nj_n}^{k_n} = \\ &= \sum_{s_1,s_2,\ldots,s_{n+1}} \mathcal{D}_{s_1,s_2,\ldots,s_{n+1}} \mathrm{e}^{\mathring{\imath} s_1\lambda_1}\mathrm{e}^{\mathring{\imath} s_2\lambda_2}\ \ldots\ \mathrm{e}^{\mathring{\imath} s_{n+1}\lambda_{n+1}}, \\ &s_1 + s_2 + \ldots + s_{n+1} = 0. \end{aligned} \tag{26}$$

All the forms of the ρ-series are trivially related to each other and, therefore, totally equivalent as far as their efficiency is concerned.

The properties of $P_i^{(0)}$ allow one to conclude that the coefficients $Q^i_{k_1k_2\ldots k_n}$ in (24) are real numbers. We use a general FFT algorithm which gives complex values for the Fourier coefficients. Therefore, actual numerical errors of $Q^i_{k_1k_2\ldots k_n}$ in (24) can be estimated from the magnitudes of the computed imaginary parts of the coefficients $Q^i_{k_1k_2\ldots k_n}$. Since we used standard double precision arithmetic we were able to compute the series (24) with an accuracy of $10^{-13} - 10^{-14}$.

The other kind of errors of the coefficients $Q^i_{k_1k_2\ldots k_n}$ as computed via the FFT is the errors of aliasing. The spectrum of $P_i^{(0)}$ is discrete, but infinite. However, when making use of the FFT in practice we should specify some finite number of harmonics to be computed, i.e. to specify the Nyquist interval of the frequencies to be considered. The larger the specified number of harmonics (that is, the higher the Nyquist frequency), the finer the grid which we should use for tabulating the function $P_i^{(0)}$. It is well known that the harmonics beyond the Nyquist interval are aliased into the Nyquist interval and spoil the computed harmonics (see, e.g., Press *et al.*, 1992). We always check that the harmonics above the noise threshold (as set from the maximal imaginary part of $Q^i_{k_1k_2\ldots k_n}$) are sufficiently far from the boundaries of the Nyquist interval. When this is not the case we increase the number of harmonics to be computed by applying FFT on a finer grid. This allows us to expect that the harmonics we compute are correct within the estimated errors.

B. We compute the expansions of the corresponding part of $p_i^{(0)}$

$$\text{a piece of } p_i^{(0)} = \sum \mathcal{A}^i_{k_1k_2\dots k_n}\rho^{k_1}_{i_1j_1}\rho^{k_2}_{i_2j_2}\,\dots\,\rho^{k_n}_{i_nj_n} \tag{27}$$

from the following relations (see, Brumberg *et al.*, 1975)

$$\mathcal{A}^i_{00\dots0} = -\frac{1}{3}Q^i_{00\dots0}, \tag{28}$$

$$\mathcal{A}^i_K = \frac{n_i^2}{(K.n)^2\left[n_i^2-(K.n)^2\right]}\times$$
$$\times\left(\left[(K.n)^2-2n_i(K.n)+\frac{3}{2}n_i^2\right]Q^i_K-\frac{3}{2}n_i^2Q^i_{-K}\right), \tag{29}$$

where $K=(k_1,k_2,\dots,k_n)$ is the multi-index, $-K=(-k_1,-k_2,\dots,-k_n)$, $(K.n)=\sum_{s=1}^N k_sn_s$, and n_i is the mean motion of the ith planet. Eq. (29) gives the coefficients $\mathcal{A}^i_{k_1k_2\dots k_n}$ as a linear combination of $Q^i_{k_1k_2\dots k_n}$ and $Q^i_{-k_1,-k_2,\dots,-k_n}$. The coefficients of these linear combinations may be quite large when the mean motions of the planets are close to commensurability. This amplifies numerical errors in $Q^i_{k_1k_2\dots k_n}$. We account for this circumstance when estimating the numerical errors of $\mathcal{A}^i_{k_1k_2\dots k_n}$ by multiplying the estimate of errors for $Q^i_{k_1k_2\dots k_n}$ by the largest encountered value of the linear coefficients in (29).

C. We compute the corresponding pieces of the τ-series

$$\text{a piece of } p_i^{(0)} = \sum \mathcal{B}^i_{k_1k_2\dots k_n}\tau^{k_1}_{i_1j_1}\tau^{k_2}_{i_2j_2}\,\dots\,\tau^{k_n}_{i_nj_n} \tag{30}$$

by tabulating the corresponding piece of $p_i^{(0)}$ (computed with the ρ-series derived at the previous step) on a grid which is uniform with respect to $\tau_{i_sj_s}$, and making use of the FFT. Again the numerical errors of $\mathcal{B}^i_{k_1k_2\dots k_n}$ can be estimated in the same manner as we discussed at the step **A** of this algorithm.

The number of angular variables n for the τ-series coincides, obviously, with that for the corresponding ρ-series. However, different forms of the τ-series are no longer trivially related to each other. Although $\tau_{ji}=\tau_{ij}^{-1}$, the expression relating, say, τ_{ij}, τ_{ik} and τ_{jk} is quite complicated. Fourier series of this expression can converge slower or faster depending on values of i, j, k. This means that different forms of (30) may converge quite differently. Therefore, we should find an optimal form of the series in each case. For $n=1$, the form of series is unique: univariate series in powers of τ_{ij}. For $n=2$ there are $m=3$ sets of $\tau_{i_sj_s}$ which are not trivially related to each other, for $n=3$ the number of the sets is $m=16$, for $n=4$ one gets $m=135$, etc.

As an additional check of the results we always explicitly check that the τ-series (30) resulting from the step **C** represent the same function (accounting for the relations between ρ_{ij} and τ_{ij} given by (2)–(4), (6) and (8)) as the ρ-series (27) resulting from the step **B** within the estimated numerical errors.

We believe that the algorithm **A**–**C** is a reliable and fast way to compute both the ρ- and the τ-series for that part of the intermediary which depends on a moderate number of angular variables (say, for $n \le 3$). For a higher number of angular variables (say, for $n > 4$) the algorithm (in its current form) becomes inefficient, since it requires too much computer memory to complete the FFT at the step **A**.

3. Mutual Perturbations of Couples of Planets

Mutual perturbations of couples of planets are described by the terms $\underset{k}{T}{}^{(i)}_{j}$ in (14)–(15) and $\underset{k}{Q}{}^{(i)}_{j}$ in (17)–(18). Both the ρ- and the τ-series are univariate series ($n = 1$) in powers of ρ_{ij} or τ_{ij} respectively

$$\mu^s \underset{s}{T}{}^{(i)}_{j} = \sum_a \mathcal{A}_a \rho^a_{ij}, \tag{31}$$

$$\mu^s \underset{s}{T}{}^{(i)}_{j} = \sum_a \mathcal{B}_a \tau^a_{ij}. \tag{32}$$

In Brumberg and Klioner (1996) the ρ- and the τ-series have been already computed for the first-order intermediary. The method of calculation used in Brumberg and Klioner (1996) is quite different from the algorithm we use in the present paper, and involves the explicit expression for $\mu \underset{1}{T}{}^{(i)}_{j}$ derived in Brumberg (1994) as well as a numerical integration of some auxiliary function appearing in that explicit expression. We have repeated the calculations using the method described in Section 2 and checked that the results coincide within expected numerical inaccuracy of the two methods. This provides us with an additional check of our algorithm. We also computed the terms $\mu^k \underset{k}{T}{}^{(i)}_{j}$, for $k = 2, 3, 4$ which give the complete intermediary of the fourth order as far as the perturbations of couples of planets are concerned. We computed these terms for all 56 couples of planets (we consider 8 major planets except Pluto). The results are illustrated by Tables I–IV and Figure 1.

The principal conclusions are that the τ-series are more efficient than the ρ-series when we consider mutual perturbations of close couples of planets (when the ratio of the semi-major axes a_i/a_j is close to

Table I. Number of harmonics for the closest couple Venus–Earth ($a_2/a_3 \approx 0.723$). The intermediary for this pair involves maximal number of harmonics. We give the number of harmonics in the ρ-series (left values) and the τ-series (right values) whose magnitude is higher than ε, for $\varepsilon = 10^{-6}, 10^{-7}, \ldots, 10^{-15}$. The terms of the first four orders has been computed. Last row contains the number of harmonics for the sum of the terms of all four orders. The number of harmonics for the sum is almost the same as for the first-order intermediary since the range of harmonics of different orders almost totally overlap. For this pair the τ-series give maximal compression as compared to the ρ-series. See, also Figure 1.

Order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}
1	8/16	14/20	22/24	30/28	39/32	48/35	58/38	66/42	83/46	96/48
2	0/ 0	0/ 0	0/ 0	1/ 0	8/14	16/26	24/32	32/36	42/42	55/46
3	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	5/ 7	13/16	19/27
4	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0	0/ 0
Σ	8/16	14/20	22/24	30/28	39/32	48/34	58/38	66/42	83/46	96/48

1) and/or when a relatively high precision is required. It is the mutual perturbations of close couples (primarily Venus–Earth and Jupiter–Saturn) which present the most difficult task when constructing an analytical theory of planetary motion. For these couples the τ-series represent the intermediary in a more compact form. However, even for the closest couple Venus–Earth and for a very high accuracy of 10^{-15} the ratio of the terms required for the ρ-series and the τ-series is not very large: $96/48 = 2$ (see, the first row of Table I). Moreover, for the couples with $\min(a_i/a_j, a_j/a_i) \ll 1$ or if a relatively low accuracy is required (even for close couples), the ρ-series are usually more preferable. These conclusions are valid both for the first-order terms and for the higher-order ones.

This result is in general agreement with the results exposed in Chapront and Simon (1988, 1996): the expansions of elliptic functions give a considerable advantage when mutual perturbations of two planets are considered.

Table II. Number of harmonics for the couple Uranus–Neptune ($a_7/a_8 \approx 0.638$). The structure of the table is the same as that of Table I. For this pair of planets the intermediary has the largest magnitude ($\sim 4.3 \cdot 10^{-3}$). It is evident from the table that the compression factor (that is the ratio of the number of terms in the corresponding pieces of the ρ-series and the τ-series) for this pair is lower than for the pair Venus–Earth. We see that this compression factor becomes lower for the higher-order terms. It is also clear from the table that due to close commensurability of mean motions of the planets ($n_7/n_8 \approx 1.961 \sim 2$) the expansion (13) of the intermediary $p_i^{(0)}$ in powers of μ converges in some cases slower than one could expect apriori: the value of the formal small parameter in this case is $\mu\kappa_{78} = 5.2 \cdot 10^{-5}$ while the typical ratio of terms of subsequent orders is $> 10^{-2}$. One can expect that the fifth-order term $\underset{5}{T}{}_8^{(7)}$ (which we did not consider) for this pair should be of order of $10^{-9} - 10^{-10}$ and should be accounted for if this level of accuracy is to be attained.

Order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}
1	11/18	17/22	22/24	30/28	37/31	43/34	51/37	58/40	68/43	79/46
2	6/11	9/18	12/22	17/26	22/30	29/34	39/38	48/41	56/44	65/48
3	0/ 0	2/ 7	8/13	12/19	15/23	19/29	24/35	28/39	34/44	40/48
4	0/ 0	0/ 0	1/ 1	6/ 9	10/16	12/20	19/27	25/36	32/41	42/46
Σ	11/18	17/22	22/24	30/27	37/29	43/34	51/38	57/40	69/44	79/48

Table III. Number of harmonics for the couple Saturn–Jupiter ($a_5/a_6 \approx 0.544$). The structure of the table is the same as that of Table I. The data show that the compression factor for this pair is lower than for the pairs Venus–Earth and Uranus–Neptune.

order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}
1	9/11	15/13	21/16	26/19	32/21	38/23	46/26	52/29	59/31	65/34
2	3/ 4	6/ 7	10/12	14/15	22/18	28/22	36/24	42/28	49/30	56/34
3	0/ 0	1/ 1	4/ 4	6/ 7	11/11	17/13	23/18	30/23	38/27	44/31
4	0/ 0	0/ 0	0/ 0	1/ 1	4/ 4	6/ 6	10/ 9	17/15	23/17	30/21
Σ	9/11	15/13	21/16	26/19	32/21	38/24	46/27	52/29	59/32	65/35

Table IV. Total number of harmonics for all 56 pairs of planets. The data allow to look at the "averaged" situation. The structure of the table is the same as that of Table I. It is easy to see that if we consider the whole problem of 8 planets the τ-series become more efficient only if a relatively high accuracy is required (10^{-10} and higher).

order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}
1	114/164	221/263	344/374	476/486	619/592	775/695	939/791	1110/ 890	1296/ 984	1494/1086
2	14/ 26	28/ 48	54/ 75	100/120	173/198	260/288	383/378	523/ 488	679/ 607	859/ 721
3	0/ 0	5/ 12	19/ 30	31/ 45	50/ 65	82/ 92	127/143	193/ 206	280/ 283	371/ 362
4	0/ 0	0/ 0	2/ 2	9/ 16	22/ 33	32/ 46	52/ 66	80/ 92	119/ 124	176/ 167
Σ	114/164	221/263	344/374	476/485	619/590	775/695	939/793	1109/ 890	1296/ 986	1494/1089

4. Mutual Perturbation of Triplets of Planets

Mutual perturbations of a triplet of planets are described by the terms $\underset{k}{T}{}^{(i)}_{jk}$ in (15) and $\underset{k}{Q}{}^{(i)}_{jk}$ in (18). Both the ρ- and the τ-series are double Fourier series ($n = 2$ in (24), (27) and (30)).

Using the algorithm described in Section 2 we computed the second- and the third-order terms $\mu^2 \underset{2}{T}{}^{(i)}_{jk}$ and $\mu^3 \underset{3}{T}{}^{(i)}_{jk}$ for all 168 triplets of 8 major planets (except for Pluto). We computed the ρ-series

$$\mu^s \underset{s}{T}{}^{(i)}_{jk} = \sum_{a,b} \mathcal{A}_{ab} \rho^a_{ij} \rho^b_{ik} \qquad (33)$$

as well as the three form of the τ-series

$$\mu^s \underset{s}{T}{}^{(i)}_{jk} = \sum_{a,b} \mathcal{B}_{ab} \tau^a_{ij} \tau^b_{ik}, \qquad (34)$$

$$\mu^s \underset{s}{T}{}^{(i)}_{jk} = \sum_{a,b} \mathcal{B}'_{ab} \tau^a_{ij} \tau^b_{jk}, \qquad (35)$$

$$\mu^s \underset{s}{T}{}^{(i)}_{jk} = \sum_{a,b} \mathcal{B}''_{ab} \tau^a_{ik} \tau^b_{jk}. \qquad (36)$$

Directly comparing the τ-series in the forms (34)–(36) we find the most optimal one for each triplet. Tables V–VI and Figures 2-3 illustrate the structure and the efficiency of the ρ-series and the most effective form of the τ-series for two representative triplets Venus–Earth–Mars and Saturn–Jupiter–Uranus.

We have checked that our results for the ρ-series are in good agreement with the results of Brumberg, Evdokimova and Skripnichenko (1975). Small discrepancies are, probably, due to different initial values for the semi-major axes a_i, the mean motions n_i and the masses of the planets m_i.

Our calculations allows us to conclude that the τ-series are by no means more compact that the ρ-series in the case of double Fourier series (33)–(36). On the contrary, we have found that the ρ-series is typically 2–10 times more compact than the most effective form of the τ-series. The opposite situation can be found only if the double ρ-series can be effectively considered as univariate series: say, $|\mathcal{A}_{ab}| < \varepsilon$ for $b \neq 0$. An example is the triplet Neptune–Mercury–Uranus. However, these terms obviously present no difficulties already in the form of ρ-series since their amplitude is very small.

An interesting comment can be made on the appearance of resonant terms in the ρ- and τ-series. On Figure 2 the set of harmonics $\mathcal{A}_{-3k,4k}$,

$k \in Z\backslash\{0\}$ is clearly seen as squares which are much darker than their vicinity. The set of harmonics corresponds the zero-order resonance induced by the close commensurability of the mean motions of Venus, Earth and Mars: $(n_2 - n_3)/(n_3 - n_4) \approx 1.3357 \sim 4/3$. On the contrary, the corresponding τ-series on Figure 3 do not have such resonant terms. These resonant terms seem to play a positive role for compactness of the ρ-series. If we retain on Figure 4 only terms $|\mathcal{A}_{ab}| \geq 10^{-8}$, we have only two resonant terms $\mathcal{A}_{-3,4} \approx -3.23 \cdot 10^{-7}$ and $\mathcal{A}_{3,-4} \approx 3.24 \cdot 10^{-7}$ which are about 30 times larger than the cut level (these two harmonics are seen as two black squares on Figure 2). However, if we confine ourselves by the same accuracy for the τ-series $|\mathcal{B}'_{ab}| \geq 10^{-8}$, we get two blobs of significant harmonics approximately centered at the places where the two terms of the ρ-series reside and containing as many as 116 terms. These two blobs are seen on Figure 2 as two darkest areas. The largest term here has the magnitude $\mathcal{B}'_{-2,2} \approx 1.0 \cdot 10^{-7}$. One can say that, although resonant terms in ρ-series lead to practical difficulties when constructing analytical theories, they play a positive role by absorbing significant spectral power and, in a sense, make the resulting Fourier series more compact.

In principle one could suggest one more form of the τ-series for $\mu^2 \underset{2}{T}{}^{(i)}_{jk}$ and $\mu^3 \underset{3}{T}{}^{(i)}_{jk}$ which might be apriori thought to be more compact than any of the three forms (34)–(36). As we noted above the expansion of, say, τ_{ij} in terms of τ_{ik} and τ_{jk} converges very slowly for some values of i, j and k. On the other hand, Eq. (21) for $\underset{2}{Q}{}^{(i)}_{jk}$, being the right-hand side of the differential equation for $\underset{2}{T}{}^{(i)}_{jk}$ analogous to (10), contains three groups of terms which can be symbolically represented as 1) $S_{ij} \times S_{ik}$, 2) $S_{ij} \times S_{jk}$, and 3) $S_{ik} \times S_{jk}$, where S_{ab} is a part of the first-order intermediary related to the couple (a, b) or function $\psi^{(ab)}_{\dots}$. Each factor S_{ab} can be expanded in the ρ- or the τ-series in powers of ρ_{ab} or τ_{ab} respectively. Therefore, each of the three group has its "natural" form of the τ-series: (34) for the first group, (35) for the second group, and (36) for the third one. Using this "natural" forms of the τ-series we avoid the implicit use of the expansions of τ_{ij} in terms of τ_{ik} and τ_{jk} and similar ones, that may improve the convergence. We have shown in Section 3 that for the factors S_{ab} in many cases the τ-series are more compact than the ρ-series. The same is true for the expansions of $\psi^{(ab)}_{\dots}$. Therefore, the "natural" τ-series should give also more compact representation for the three parts of $\underset{2}{Q}{}^{(i)}_{jk}$. Explicit calculations, which we have done for two triplets Venus–Earth–Mars and Saturn–Jupiter–Uranus, shows that it is really true: the τ-series

Table V. Number of harmonics in the ρ-series (left values) and the most effective form of the τ-series (right values) for $\mu^2 \underset{2}{T}{}^{(2)}_{34}$ and for $\mu^3 \underset{3}{T}{}^{(2)}_{34}$ for the triplet Venus–Earth–Mars. The most efficient form of the τ-series for this triplet is Eq. (35). We give the number of harmonics whose absolute values are larger than ε. We see that for any ε the ρ-series are more compact than the τ-series.

order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}
2	0/ 0	2/ 1	2/116	8/246	20/449	55/710	127/1074	244/1516
3	0/ 0	0/ 0	0/ 0	0/ 0	2/ 0	10/107	29/ 283	71/ 468

of the "natural" form are more compact for each of the three parts of $\underset{2}{Q}{}^{(i)}_{jk}$ than the corresponding ρ-series. However, since we have now three different kinds of the τ-series (instead of one in case of any of the forms (34)–(36)), and these three τ-series cannot be combined into one double Fourier series, it does not even mean that such a "natural" representation of $\underset{2}{Q}{}^{(i)}_{jk}$ is the most compact one. For example, for the triplet Venus–Earth–Mars the three "natural" τ-series for $\mu^2 \underset{2}{Q}{}^{(i)}_{jk}$ have in total 1769 harmonics with magnitude larger than 10^{-18} (this approximately corresponds to the resulting accuracy of 10^{-13} for $\mu^2 \underset{2}{T}{}^{(i)}_{jk}$, since for the triplet under consideration resonances multiply errors in $\mu^2 \underset{2}{Q}{}^{(i)}_{jk}$ by a factor of 10^5), the corresponding ρ-series contain as many as 2945 harmonics, and the most compact form of the τ-series turns our to be (34) and contains only 1193 terms at the same cut level. On the other hand, the "natural" form of the τ-series for the intermediary itself $\mu^2 \underset{2}{T}{}^{(i)}_{jk}$ is again much longer than the ρ-series even for the three parts of $\mu^2 \underset{2}{T}{}^{(i)}_{jk}$ generated by the corresponding three parts of $\mu^2 \underset{2}{Q}{}^{(i)}_{jk}$. For the triplet Venus–Earth–Mars we have only 244 harmonics in the ρ-series with magnitudes larger than 10^{-13} (see, Table V and Figure 2) while the "natural" τ-series contain in total 2860 terms, and the most compact form of the τ-series, which turns out to be (35) in this case, 1516 terms (see, Table V and Figure 3). The situation for the triplet Saturn–Jupiter–Uranus is analogous. Thus, the "natural" form of the τ-series are by no means more compact for the intermediary than the forms (34)–(36) and our conclusion on the compactness of the ρ- and τ-series formulated above remains valid.

Table VI. Number of harmonics in the ρ-series (left values) and the most effective form of the τ-series (right values) for $\mu^2 \underset{2}{T}{}^{(6)}_{57}$ and for $\mu^3 \underset{3}{T}{}^{(6)}_{57}$ for the triplet Saturn–Jupiter–Uranus. The most efficient form of the τ-series for this triplet is Eq. (34). Again we see that for any ε the ρ-series are more compact than the τ-series.

order \ ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	$2 \cdot 10^{-13}$
2	1/ 0	6/27	24/72	47/179	96/284	165/453	259/706	320/860
3	0/ 0	0/ 0	3/ 5	13/ 45	35/ 98	78/191	149/334	223/519

5. Concluding Remarks

Our experience with computing the expansions for intermediate orbit for general planetary theory shows that in principle it is nowadays quite feasible task to construct the whole theory with reasonable accuracy in a semi-analytical form on a workstation-class computer (HP9000/715 and Sun Ultra 1 have been used for the calculations described in the present paper). The approach suggested in Brumberg (1979) and Brumberg and Chapront (1973) together with a numerical algorithm similar to steps **A** and **B** described in Section 2 could be used herewith. Although it is clear that the problems of convergence of the series in μ as well as in eccentricities and inclinations require more detailed study and may even make practical value of such a theory questionable.

As for the elliptic function approach we can summarize our results as follows. The expansions based on elliptic functions do give certain level of compression of the resulting series, when we deal with mutual perturbations of a pair of planets. However, the level of compression is not very high (<2 for actual major planets) and depends on the considered precision (the lower the required precision, the lower the advantage of the elliptic function expansions). These results are in good agreement with the conclusions published previously by Chapront and Simon (1988, 1996), and Brumberg and Klioner (1996). However, the τ-series approach does not give any compression in case of mutual perturbations of a triplet of planets when we should consider multivariate Fourier series. On the contrary, the ρ-series turn out to be more efficient than the most compact form the τ-series. We have also computed the triple Fourier series for $\mu^3 \underset{3}{T}{}^{(i)}_{jkl}$ for a few sets of four planets. For these terms the conclusion is the same as for $\mu^k \underset{k}{T}{}^{(i)}_{jk}$: the ρ-series are more compact than the most effective form of the τ-series. This allows us to conclude that the elliptic function expansions (at least in the forms considered in this paper) have no practical interest when applied to the complete problem of N planets.

It may happen that some other kind of expansions cure the problem and allows to compress multivariate Fourier series of the GPT. The numerical techniques which we used in this research and the corresponding software allow one to test quickly other ideas aimed at a compression of the GPT-like theories of motion which may be proposed in the future.

Let us also mention certain parallels between the compression of planetary theories and the image compression techniques developed in the field of signal processing. The principal problem in constructing general planetary theory is to find a convenient (compact) representation of the functions $\underset{2}{T}{}^{(i)}_{jk}$, $\underset{3}{T}{}^{(i)}_{jk}$, $\underset{3}{T}{}^{(i)}_{jkl}$, etc. These functions are known to be periodic functions of several variables (two variables for $\underset{2}{T}{}^{(i)}_{jk}$ and $\underset{3}{T}{}^{(i)}_{jk}$, three variables for $\underset{3}{T}{}^{(i)}_{jkl}$, etc.). In order to compute these functions with a given accuracy we can tabulate each of them on a sufficiently fine grid of the corresponding dimension. These tabulated values may be considered as [multidimensional] images. What we are aimed at is to find compact representations of these images allowing to compute the original functions as accurate and as quickly as possible. One of the approaches is to calculate the Fourier expansions of the functions and to retain only those coefficients which are larger (by absolute value) than some specified cut level. This is the standard way of constructing planetary theories in a purely trigonometric form. The elliptic function approach consists in a change of independent variables and applying the same Fourier transformation. From the point of view of image compression the Fourier transformation is one of the most popular method of compression, but not the only possible one. On the contrary, there are many other transformations which use another [non-trigonometrical] basic functions and which gives in many cases better results for image compression than the Fourier transformation. Wavelet transform is an example, but by no means the only possible one (see, e.g., Hunt, 1978; Press, *et al.*, 1992). Although the alternative compression techniques cannot be immediately applied to constructing planetary theories (most of compression techniques have been tested on images with low “dynamical range”, say only 256 possible values of “brightness”), investigations in this area may lead, in our opinion, to valuable results.

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References

Brumberg V.A., 1966. Representation of the Coordinates of the Planets by Trigonometric Series. *Transactions of the ITA* **11**, 3

Brumberg V.A., 1970. Application of Hill's Lunar Method in General Planetary Theory. In: G.E.O. Giacaglia (ed.), *Periodic Orbits, Stability and Resonances*, Reidel, Dordrecht, 410

Brumberg V.A., 1992. General Planetary Theory Revisited with the Aid of Elliptic Functions. In: H.Kinoshita and H.Nakai (eds.), Proc. of the 25th symposium on Celestial Mechanics, National Astronomical Observatory, Tokyo, 156

Brumberg V.A., 1994. General Planetary Theory in Elliptic Functions. *Celestial Mechanics*, **59**, 1

Brumberg V.A., 1996. Theory Compression with Elliptic Functions. In: S.Ferraz-Mello, B.Morando, J.E.Arlot (eds.), *Dynamics, ephemerides and astrometry in the solar system*, Kluwer, Dordrecht, 89

Brumberg V.A., Chapront J., 1973. Construction of a General Planetary Theory of the First Order. *Celestial Mechanics*, **8**, 335

Brumberg V.A., Evdokimova L.S., Skripnichenko V.I., 1975. Quasi-periodic Intermediate Orbits of Major Planets and Resonances of the Zero Order. Astronomicheskij Zhurnal, **52**, 420 (in Russian)

Brumberg V.A., Evdokimova L.S., Skripnichenko V.I., 1978. Mathematical Results of the General Planetary Theory in Rectangular Coordinates. In: V.Szebehely (ed.), *Dynamics of Planets and Satellites and Theories of Their Motion*, Reidel, Dordrecht, 33

Brumberg V.A., Klioner S.A., 1996. Numerical Efficiency of the Elliptic Function Expansions of the First-Order Intermediary for General Planetary Theory. In: S.Ferraz-Mello, B.Morando, J.E.Arlot (eds.), *Dynamics, ephemerides and astrometry in the solar system*, Kluwer, Dordrecht, 101

Chapront J., Simon J.L., 1988. Perturbations de premier ordre pour les couple de planètes. *unpublished manuscript*

Chapront J., Simon J.L., 1996. Planetary Theories with the Aid of Expansions of Elliptic Functions. *Celestial Mechanics*, **63**, 171

Hunt B.R., 1978. Digital Image Processing. In: A.V.Oppenheim (ed.), *Applications of Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, New Jersey

Klioner S.A., 1992. Some typical algorithms of the perturbation theory within Mathematica and their analysis. In: H.Kinoshita and H.Nakai (eds.), Proc. of the 25th symposium on Celestial Mechanics, National Astronomical Observatory, Tokyo, 172

Press W.H., Teukolsky S.A., Vetterling W.T., Flannery B.P., 1992. Numerical Recipes In Fortran: the art of scientific computing. Cambridge University Press, New York

Richardson D.L., 1982. A Third–Order Intermediate Orbit for Planetary Theory. *Celestial Mechanics*, **26**, 187

Williams C.A., Van Flandern T. and Wright E.A. 1987. First Order Planetary Perturbations with Elliptic Functions. *Celestial Mechanics*, **40**, 367
Williams C.A. 1992. A Planetary Theory with Elliptic Functions and Elliptic Integrals Exhibiting No Small Divisors. In: S. Ferraz-Mello (ed.), *Chaos, Resonance and Collective Dynamical Phenomena in the Solar System*, 43, Kluwer, Dordrecht

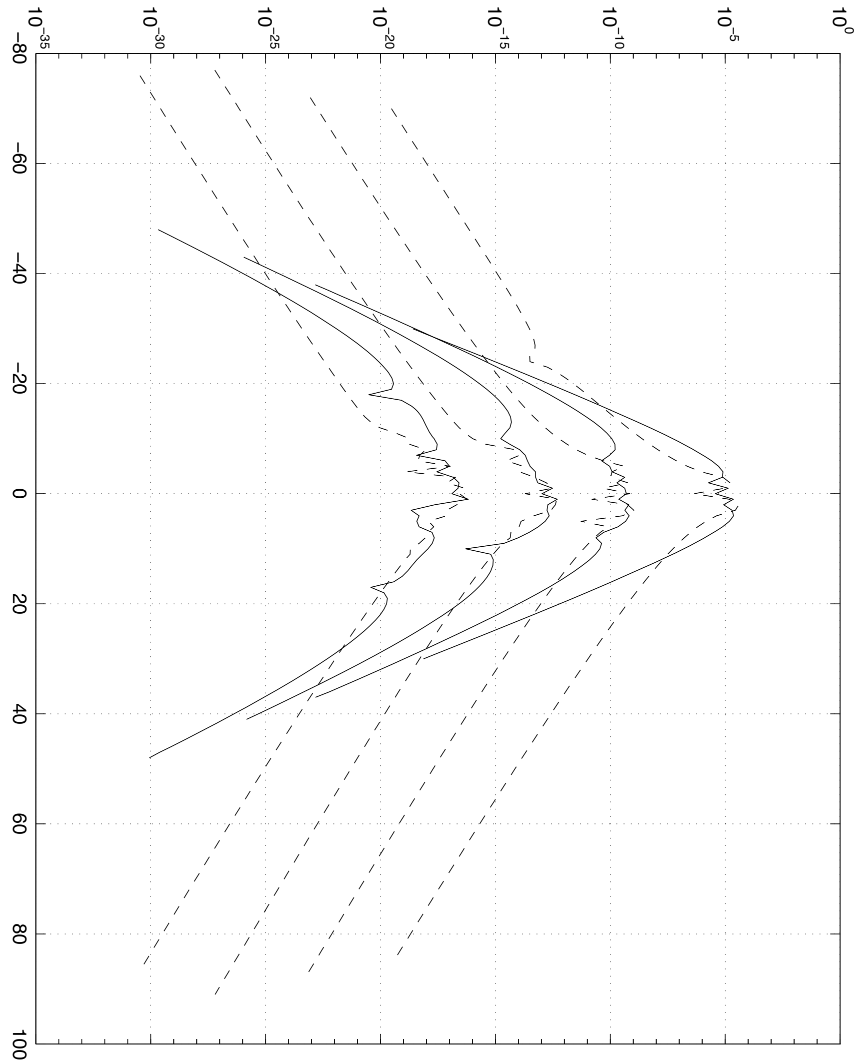

Figure 1. Absolute values of the coefficients $\mathcal{A}_a$ of the ρ-series (dashed lines) and the coefficients $\mathcal{B}_a$ of the τ-series (solid lines) as functions of a for the couple Venus–Earth. The terms of the 1st, 2nd, 3rd and 4th orders are displayed separately on the same plot. At any order for some a values of $|\mathcal{B}_a|$ become lower than $|\mathcal{A}_a|$ and continue to decrease faster than $|\mathcal{A}_a|$. This general behavior is typical for all pairs of planets. The advantages of the τ-series in this case are obvious from the Figure. See, also Table I.

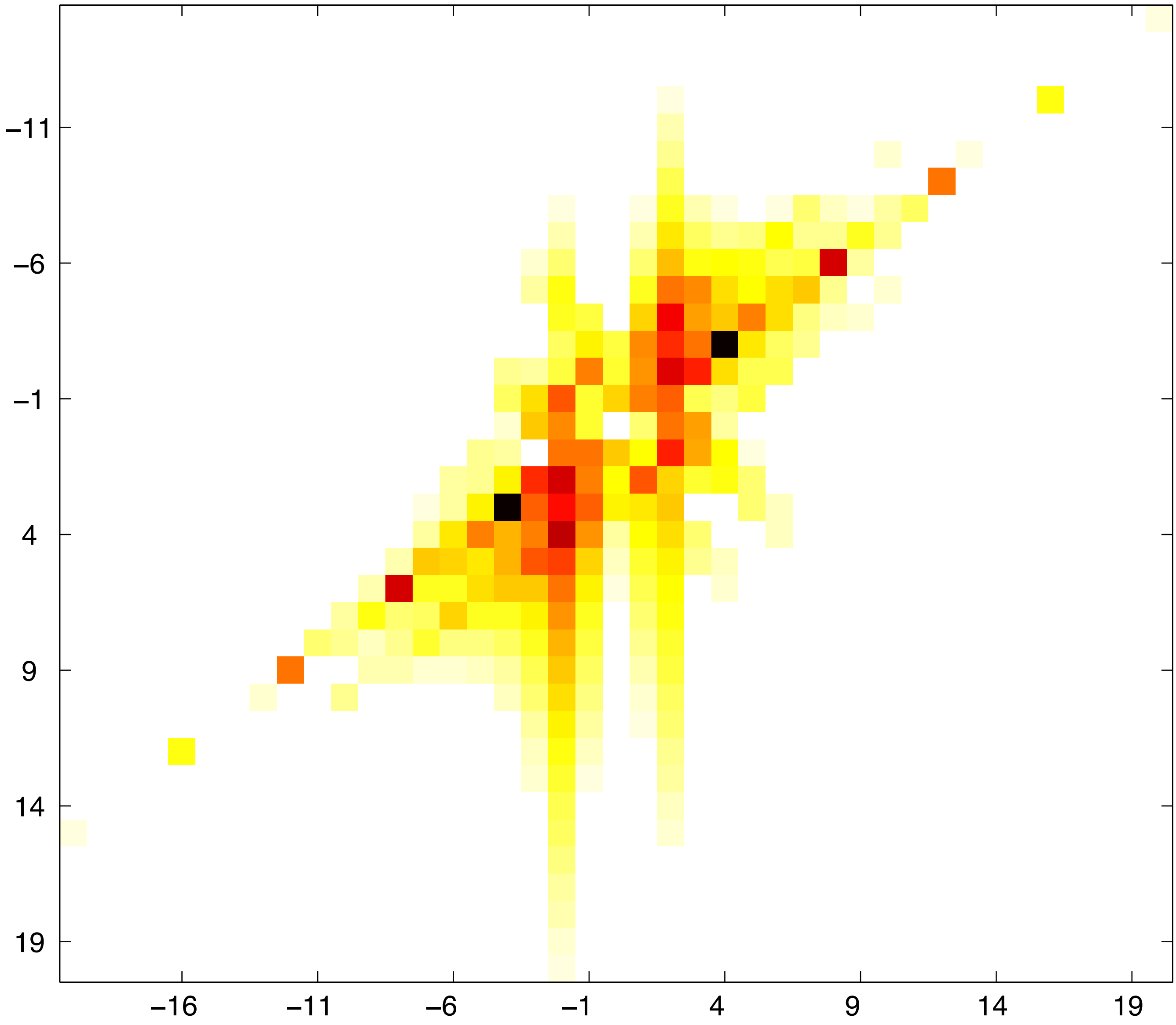

Figure 2. The coefficients $\mathcal{A}_{ab}$ of the ρ-series $\sum_{a,b} \mathcal{A}_{ab}\rho_{23}^a\rho_{34}^b$ for the second-order term $\mu^2 \underset{2}{T}{}^{(2)}_{34}$ for the triplet Venus–Earth–Mars. We present the double Fourier series as an image. Each coefficient is displayed as a square of different gray level. The larger the absolute value of the coefficient, the darker the square. The abscissas show the values of b, while the ordinates show the values of a. All the harmonics $|\mathcal{A}_{ab}| \geq 10^{-13}$ are shown on the Figure. Total number of harmonics is 244. See, also Table V.

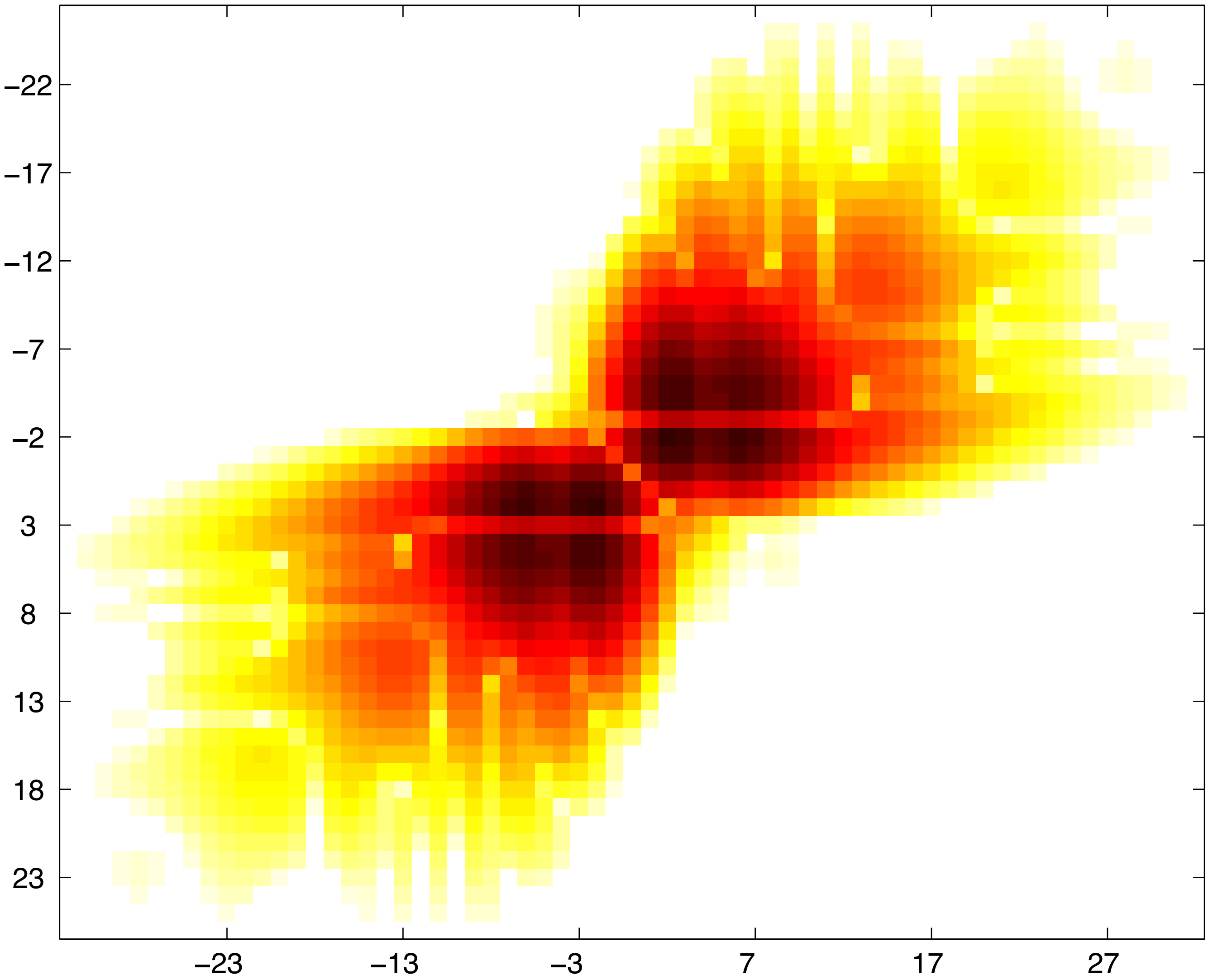

Figure 3. The coefficients $\mathcal{B}'_{ab}$ of the τ-series $\sum_{a,b} \mathcal{B}'_{ab} \tau_{23}^a \tau_{34}^b$ for the second-order term $\mu^2 T^{(2)}_{2\ 34}$ for the triplet Venus–Earth–Mars. This form of the τ-series is the most compact one for this triplet. The graphical representation of the harmonics is the same as on Figure 2. All the harmonics $|\mathcal{B}'_{ab}| \geq 10^{-13}$ are shown on the Figure. Total number of harmonics is 1516. We see that the τ-series are by far less compact than the ρ-series in this case (cf. Figure 2). See, also Table V.